Wahrscheinlichkeitstheorie
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1. Mathemathischer Gerüst

Der Tripel $(\Omega, \mathcal{F}, \mathbb{P})$ heisst ein Wahrscheinlichkeitsraum.

Grundraum: Ω

Menge aller mögliche Ergebnisse (outcomes) $\omega \in \Omega$.

$\Omega = \{\omega_1, \dots, \omega_i\}$

Bsp: $\Omega = \{\text{Kopf}, \text{Zahl}\}$ $\quad \Omega = \{0, 1\}$

$\Omega = \{1, 2, \dots, 6\}$

$\Omega = \{1, 2, \dots, 6\}^3 \to \omega = (\omega_1, \omega_2, \omega_3) \in \Omega$

$\Omega = [0,1]^2$

$\mathcal{P}(\cdot)$ Potenzmenge (Menge aller Teilmenge von Ω)

Menge aller Ereignisse: $\mathcal{F}$

> $\mathcal{F} \subset \mathcal{P}(E)$ mit folgende Bedingungen:
> H1. $\Omega \in \mathcal{F}$
> H2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
> H3. $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

Bem: eine Menge $\mathcal{F} \subset \mathcal{P}(E)$ die alle 3 Bedingungen erfüllt heisst ein σ-algebra

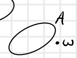
Ereigniss (Event):

> Ein Ereignis $A, B, C, \dots$ ist gegeben durch ein Teilmenge $A \subset \Omega$.

Bsp: - $A = \{2, 4, 6\}$ $\hat{=}$ "A die is Even"

- $A = [0,1] \times [1/2, 1]$ $\hat{=}$ "droplet in upper part of the square"
- $A = \{\omega \in \Omega : \omega \leq 3\}$

Sei $\omega \in \Omega$ ein Ergebnis, A ein Ereigniss:

- A geschieht falls $\omega \in A$ 
- A geschieht nicht falls $\omega \notin A$ 

$A = \emptyset$ geschieht nie, $A = \Omega$ geschieht immer.

Operation auf Ereignisse und Interpretation

Prop 1.5 (Consequences of definition) We have

i. $\emptyset \in \mathcal{F}$

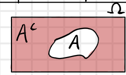
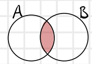
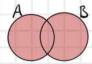
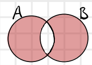
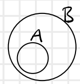
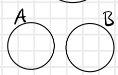
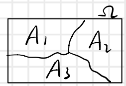
ii. $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

iii. $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$

iv. $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

Mengenoperation

Seien $A, B \in \mathcal{F}$ zwei Ereignisse

Ereigniss	Graphische Representation	Probab. Interpretation
A^c		A geschieht nicht
$A \cap B$		A und B geschehen
$A \cup B$		A oder B geschieht
$A \Delta B$		Eine und nur eine von A oder B geschieht
$A \subset B$		Wenn A geschieht, so geschieht auch B
$A \cap B = \emptyset$		A und B können nicht gleichzeitig geschehen
$\Omega = A_1 \cup A_2 \cup A_3$ mit A_1, A_2, A_3 paarweise disjunkt		$\forall$ Ergebnis ω, eine und nur eine von A_1, A_2, A_3 geschieht

De-Morgansche Gesetz

$(A \cap B)^c = A^c \cup B^c \sim \overline{\bigcap_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$

$(A \cup B)^c = A^c \cap B^c \sim \overline{\bigcup_{i \in I} A_i} = \bigcap_{i \in I} \overline{A_i}$

Inklusions-Exklusionsprinzip

Seien $A_i, i = 1, \dots, n$ beliebige Ereignisse, wobei $n \in \mathbb{N}$, es gilt dann

$$\boxed{\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq n} \mathbb{P}[A_{i_1} \cap \dots \cap A_{i_k}]}$$

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3)$

Def 1.6: Let Ω be a sample space, let $\mathcal{F}$ be a set of events. A probability measure on $(\Omega, \mathcal{F})$ is a map

$$\boxed{\begin{aligned} \mathbb{P}: \mathcal{F} &\to [0,1] \\ A &\mapsto \mathbb{P}[A] \end{aligned}}$$

that satisfies the following properties

P1: $\mathbb{P}[\Omega] = 1$

P2: $\mathbb{P}[A] = \sum_{i=1}^{\infty} \mathbb{P}[A_i]$ if $A = \bigcup_{i=1}^{\infty} A_i$ (disjoint union)

"A probability measure is a map that associates to each event a number in $[0,1]$"

Def 1.7. Let Ω be a sample space, $\mathcal{F}$ a set of event, and $\mathbb{P}$ a probability measure. The triple $\boxed{(\Omega, \mathcal{F}, \mathbb{P})}$ is called a probability space.

Prop. 1.8: Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$

i. $\boxed{\mathbb{P}[\emptyset] = 0}$

ii. Let $k \geq 1$, let $A_1, \dots, A_k$ be k pairwise disjoint events, then: $\boxed{\mathbb{P}[A_1 \cup \dots \cup A_k] = \mathbb{P}[A_1] + \dots + \mathbb{P}[A_k]}$

iii. Let A be an event, then $\boxed{\mathbb{P}[A^c] = 1 - \mathbb{P}[A]}$

iv. If A and B are two events, then: $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]$

Prop. 1.9 Let $A, B \in \mathcal{F}$, then $A \subset B \Rightarrow \mathbb{P}[A] \leq \mathbb{P}[B]$

Prop 1.10 (Union bound) Let $A_1, A_2, \dots$ be a sequence of events, then

$$\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] \leq \sum_{i=1}^{\infty} \mathbb{P}[A_i]$$

Also applies to a finite collection of events

Prop 1.12 Let A_n be an increasing sequence of events (i.e. $A_n \subset A_{n+1}$ for every n). Then $\lim_{n \to \infty} \mathbb{P}[A_n] = \mathbb{P}\left[\bigcup_{n=1}^{\infty} A_n\right]$

Let B_n be a decreasing sequence of events (i.e. $B_n \supset B_{n+1}$ $\forall n$)

Then. $\lim_{n \to \infty} \mathbb{P}[B_n] = \mathbb{P}\left[\bigcap_{n=1}^{\infty} B_n\right]$

Laplace models and counting

Def 1.14 Let Ω be a finite sample space. The Laplace model on Ω is the triple $(\Omega, \mathcal{F}, \mathbb{P})$ where:

- $\boxed{\mathcal{F} = \mathcal{P}(\Omega)}$
- $\mathbb{P}: \mathcal{F} \to [0,1]$ is defined by $\boxed{\mathbb{P}[A] = \frac{|A|}{|\Omega|}}$ $\forall A \in \mathcal{F}$

Random variables and distribution functions

Def 1.15 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable (r.v.) is a map $\boxed{X: \Omega \to \mathbb{R}}$ such that for all $a \in \mathbb{R}$

$$\boxed{\{\omega \in \Omega : X(\omega) \leq a\} \in \mathcal{F}}$$

Notation: for $a \leq b$ we write

$\{X \leq a\} = \{\omega \in \Omega : X(\omega) \leq a\}$

$\{a < X \leq b\} = \{\omega \in \Omega : a < X(\omega) < b\}$

$\{X \in \mathbb{Z}\} = \{\omega \in \Omega : X(\omega) \in \mathbb{Z}\}$

$\mathbb{P}[X \leq a] = \mathbb{P}[\{X \leq a\}] = \mathbb{P}[\{\omega \in \Omega : X(\omega) \leq a\}]$

Def 1.16 Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The distribution function of X is the function $\boxed{F_X : \mathbb{R} \to [0,1]}$ defined by

$$\boxed{F_X(a) = \mathbb{P}[X \leq a]} \quad \forall a \in \mathbb{R}$$

→ 1-4

$$\boxed{\begin{array}{l} \text{Verteilung: } \mathbb{P}[X=a] \ \forall a \\ \text{Verteilungsfunktion: } F_X(a) \quad \forall a \end{array}}$$

Prop 1.17 Let $a<b$ be two real numbers. Then $\boxed{\mathbb{P}[a < X \leq b] = F(b) - F(a)}$

Thm 1.18 Let X be a random variable on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The distribution function $F_X : \mathbb{R} \to [0,1]$ of X satisfies the following properties:

(i) F_X is non-decreasing

(ii) F_X is right continuous (i.e. $F_X(a) = \lim_{h \searrow 0} F(a+h)$ $\forall a \in \mathbb{R}$)

(iii) $\lim_{a \to -\infty} F(a) = 0$ and $\lim_{a \to \infty} F(a) = 1$

Thm 1.19 Let $F: \mathbb{R} \to [0,1]$ satisfying (i), (ii), (iii), then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a random variable $X: \Omega \to \mathbb{R}$ such that $F = F_X$

↳ So given a function F that satisfies Thm 1.18, one can always consider a r.v. on some $(\Omega, \mathcal{F}, \mathbb{P})$ such that $F = F_X$

Prop 1.20 Let $X: \Omega \to \mathbb{R}$ be a r.v. with distribution function F. Then $\forall a \in \mathbb{R}$

$$\mathbb{P}[X=a] = F(a) - F(a^-)$$

With props 1.17 & 1.20 it follows:

Wahrscheinlichkeit aus Verteilungsfunktionen

- $\mathbb{P}[X<a] = F_X(a^-)$
- $\mathbb{P}[X=a] = F_X(a) - F_X(a-0)$
- $\mathbb{P}[X \leq a] = F_X(a)$
- $\mathbb{P}[a < X \leq b] = F_X(b) - F_X(a)$
- $\mathbb{P}[a < X < b] = F_X(b^-) - F_X(a)$
- $\mathbb{P}[a \leq X \leq b] = F_X(b) - F_X(a^-)$
- $\mathbb{P}[a \leq X < b] = F_X(b^-) - F_X(a^-)$
- $\mathbb{P}[X \geq a] = 1 - \mathbb{P}[X<a]$
- $\mathbb{P}[X > a] = 1 - \mathbb{P}[X \leq a]$

1.5 Conditional probabilities

Def 1.23 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some probability space. Let A, B be two events with $\mathbb{P}[B] > 0$. The conditional probability of A given B is defined by:

$$\boxed{\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} \iff \mathbb{P}[A \cap B] = \mathbb{P}[B]\,\mathbb{P}[A|B]}$$

Rmk: - $\mathbb{P}[B|B] = 1$

- Prop 1.25: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be some proba space. Let B be an event with positive probability. Then $\mathbb{P}[\cdot|B]$ is a probability measure on Ω.

Prop 1.26 Formula of total probability.

Let $B_1, \dots, B_n$ be a partition of the sample space Ω with $\mathbb{P}[B_i] > 0$ for every $1 \leq i \leq n$. Then, one has

$$\boxed{\mathbb{P}[A] = \sum_{i=1}^{n} \mathbb{P}[A|B_i]\,\mathbb{P}[B_i] = \sum_{i=1}^{n} \mathbb{P}[A \cap B_i]} \quad \forall A \in \mathcal{F}$$

Let $\Omega = B_1 \cup \dots \cup B_n$ and B_i's be pairwise disjoint ($B_i \cap B_j = \emptyset, i \neq j$) then $B_1, \dots, B_n$ is a partition of the sample space Ω. → 6

Prop 1.27 Bayes Formula

Let $B_1, \dots, B_n \in \mathcal{F}$ be a partition of Ω with $\mathbb{P}[B_i] > 0$ for every i. For every A with $\mathbb{P}[A] > 0$, we have:

$$\boxed{\mathbb{P}[B_i|A] = \frac{\mathbb{P}[A|B_i]\,\mathbb{P}[B_i]}{\sum_{j=1}^{n} \mathbb{P}[A|B_j]\,\mathbb{P}[B_j]}} = \frac{\mathbb{P}[A \cap B_i]}{\sum_{j=1}^{n} \mathbb{P}[A \cap B_j]}$$

Monotonie $A \cap B \subseteq A$

$\mathbb{P}(A \cap B) \leq \mathbb{P}(A)$

Paarweise disjunkt $\mathbb{P}[A \cup B] = \mathbb{P}[A] + \mathbb{P}[B]$ → 5

Def 1.28 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Two events A and B are said to be independent if

$$\boxed{\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]}$$

Rmk: - If $\mathbb{P}[A] \in \{0,1\}$, then A is independent of every event, i.e. $\forall B \in \mathcal{F}$ $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$

- If an event A is independent with itself ($\mathbb{P}[A \cap A] = \mathbb{P}[A]^2$), then $\mathbb{P}[A] \in \{0,1\}$
- A is independent of B if and only if it is independent of B^c
- The three events A, B and C are independent if
  - $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$
  - $\mathbb{P}[A \cap C] = \mathbb{P}[A]\,\mathbb{P}[C]$
  - $\mathbb{P}[B \cap C] = \mathbb{P}[B]\,\mathbb{P}[C]$
  - $\mathbb{P}[A \cap B \cap C] = \mathbb{P}[A] \cap \mathbb{P}[B] \cap \mathbb{P}[C]$

"Independent events are events that do not influence each other"

Prop. 1.30 Let $A, B \in \mathcal{F}$ be two events with $\mathbb{P}[A], \mathbb{P}[B] > 0$. Then the following are equivalent:

(i) $\mathbb{P}[A \cap B] = \mathbb{P}[A]\,\mathbb{P}[B]$ (A & B independent)

(ii) $\mathbb{P}[A|B] = \mathbb{P}[A]$

(iii) $\mathbb{P}[B|A] = \mathbb{P}[B]$

Def 1.32 Let $X_1, \dots, X_n$ be n random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $X_1, \dots, X_n$ are independent if

$$\boxed{\mathbb{P}[X_1 \leq a_1, \dots, X_n \leq a_n] = \mathbb{P}[X_1 \leq a_1] \cdot \ldots \cdot \mathbb{P}[X_n \leq a_n]} \quad \forall a_1, \dots, a_n \in \mathbb{R}$$

Def 1.33 Let $X_1, X_2, \dots$ be an infinite sequence of r.v.. $X_1, X_2, \dots$ are independent if $X_1, \dots, X_n$ are independent for every n.

Thm 1.34 Let $F_1, \dots, F_n$ be n distribution functions. Then there exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and n random variables $X_1, \dots, X_n$ on this probability space such that

- $\forall i$, X_i has distribution function F_i, and
- $X_1, \dots, X_n$ are independent

Def 1.35 Some random variables $X_1, X_2, \dots$ are said to be independent and identically distributed (iid) if

- they are independent
- they have the same distribution function, i.e. $\forall i,j$ $F_{X_j} = F_{X_i}$

Discrete distributions

Preliminaries: infinite sums.

Let E be finite or countable. The sum $\sum_{x\in E} a_x$ is:

- always well defined if E is finite
- not always well defined if E is countable

But if it is countable, the existance of a sequence $F_n : \forall n, F_n \subset E$ and $F_{n+1} \subset E$ is guaranteed.

$\cdot\ E = \bigcup_{n\in\mathbb{N}} F_n$

Def 2.1 Let $(a_x)_{x\in E}$ be a sequence of nonnegative numbers (i.e. $a_x \geq 0\ \forall x$). The sum of the a_x is defined by

$$\sum_{x\in E} a_x := \sup_{\substack{F\subset E \\ F \text{ finite}}} \sum_{x\in F} a_x$$

Rmk: - The result can be infinite

- $\lim_{n\to\infty} \left(\sum_{x\in F_n} a_n\right) = \sum_{x\in E} a_x$

Def 2.3 A sequence $(a_x)_{x\in E}$ of real numbers is said to be integrable if

$$\sum_{x\in E} |a_x| < \infty$$

Thm 2.5-2.6 (Fubini's theorem) we can always permute the sums when all the terms are nonnegative. If the terms are not all nonnegative, one needs an integrability condition, then we can permute the sums.

Definitions

Def 1.21 A random variable is said to be discrete if its image

$$X(\Omega) = \{x\in\mathbb{R} : \exists \omega\in\Omega\ X(\omega) = x\}$$

is at most countable.

I.e. if X only takes finitely or countably many values $x_1, x_2, \dots$

Def 2.7 Equivalent: there exists some set $E\subset\mathbb{R}$ finite or countable such that

$$X(\omega)\in E \qquad \forall \omega\in\Omega$$

Def 2.8 Let X be a discrete r.v. taking values in some finite or countable set $E\subset\mathbb{R}$. The distribution of X is the sequence of numbers $(p_x)_{x\in E}$ defined by:

$$\boxed{\forall x\in E \quad p_x = \mathbb{P}[X=x]}$$

Prop. 2.9 The distribution p_x of a discrete r.v. satisfies $\boxed{\sum_{x\in E} p_x = 1}$

Conversly if we are given a sequence p_x with values in $[0,1]$ s.t. $\sum_{x\in E} p_x = 1$, then $\exists$ a proba. space $(\Omega, \mathcal{F}, \mathbb{P})$ and an r.v. X with dist. p_x

Bernoulli random variables

Let $0\leq p\leq 1$. A random variable X is said to be a Bernoulli r.v. with parameter p if it takes values in $E=\{0,1\}$ and

$$\boxed{\mathbb{P}[X=0] = 1-p \quad \text{and} \quad \mathbb{P}[X=1] = p} \qquad X\sim \mathrm{Ber}(p)$$

Binomial random variable

Let $0\leq p\leq 1$, let $n\in\mathbb{N}$. A random variable X is said to be a binomial random variable with parameters n and p if it takes values in $E=\{0,\dots,n\}$ and

$$\boxed{\mathbb{P}[X=k] = \binom{n}{k} p^k (1-p)^{n-k}, \quad \forall k\in\{0,\dots,n\}} \qquad X\sim \mathrm{Bin}(n,p)$$

Prop 2.12 (Sum of independant Bernoulli) Let $0\leq p\leq 1$, $n\in\mathbb{N}$, $X_1,\dots,X_n$ be iid Bernoulli r.v. with parameter p. Then

$$\boxed{S_n := X_1 + \dots + X_n}$$

S_n is a binomial variable with parameter n and p.

↳ $\mathrm{Bin}(1,p)$ is the same distribution as $\mathrm{Ber}(p)$

↳ if $X\sim\mathrm{Bin}(m,p)$ and $Y\sim\mathrm{Bin}(n,p)$ and X, Y are independent then $X+Y\sim\mathrm{Bin}(m+n,p)$ $\longrightarrow 8$

Geometric random variable

Let $0 < p \leq 1$. A r.v. X is said to be a geometric random variable with parameter p if it takes values in $E=\mathbb{N}\setminus\{0\}$ and

$$\boxed{\forall k\in\mathbb{N}\setminus\{0\} \quad \mathbb{P}[X=k] = (1-p)^{k-1}\cdot p} \qquad X\sim\mathrm{Geom}(p)$$

→ Appears as the first sucess in an infinite sequence of Bernoulli experiments with parameter p. Or, more formally:

Prop 2.15 Let $X_1, X_2, \dots$ be a sequence of infinitely many independent Bernoulli r.v.'s with parameter p. Then

$$T := \min\{n\geq 1 : X_n = 1\}$$

is a geometric random variable with parameter p.

Prop 2.17 (Absence of memory of the geometric distribution) Let $T\sim\mathrm{Geom}(p)$ for some $0<p<1$. Then

$$\boxed{\mathbb{P}[T\geq n+k \mid T > n] = \mathbb{P}[T\geq k]} \qquad \forall n\geq 0\ \forall k\geq 1$$

Poisson random variable

Let $\lambda > 0$ be a positive real number. A r.v. X is said to be a Poisson random variable with parameter λ if it takes values in $E=\mathbb{N}$ and

$$\boxed{\forall k\in\mathbb{N} \quad \mathbb{P}[X=k] = \frac{\lambda^k}{k!} e^{-\lambda}}$$

↳ Appears as an approximation of a binomial distribution when n is large and p is small. Formally:

Prop 2.19 (Poisson approximation of the binomial) Let $\lambda > 0$. For every $n\geq 1$ consider a random variable $X_n\sim\mathrm{Bin}(n, \frac{\lambda}{n})$. Then

$$\forall k\in\mathbb{N} \quad \lim_{n\to\infty} \mathbb{P}[X_n = k] = \mathbb{P}[N=k],$$

where N is a Poisson r.v. with parameter $\lambda = p\cdot n$

2.3 Joint distribution and image of random variables

Def 2.21 Let $X_1:\Omega\to E_1, \dots, X_n:\Omega\to E_n$ be n discrete r.v.s on $(\Omega,\mathcal{F},\mathbb{P})$ with values in some finite or countable sets $E_1,\dots,E_n$. The family $(p_{x_1,\dots,x_n})_{x_1\in E_1,\dots,x_n\in E_n} = \mathbb{P}[X_1=x_1,\dots,X_n=x_n]$

is called the joint distribution of the r.v's $X_1,\dots,X_n$

↳ Characterizes the probabilistic properties of the random vector $(X_1,\dots,X_n)$

- If $X_1,\dots,X_n$ are independant, then:

$$p_{x_1,\dots,x_n} = \mathbb{P}[X_1=x_1]\cdot\ldots\cdot\mathbb{P}[X_n=x_n]$$

Prop 2.22 Let $n\geq 1$ and $\phi:\mathbb{R}^n\to\mathbb{R}$ be an arbitrary function. Let $X_1:\Omega\to E_1,\dots,X_n:\Omega\to E_n$ be n discrete r.v. on $(\Omega,\mathcal{F},\mathbb{P})$ with values in some finite or countable sets $E_1,\dots,E_n$. Then $Z=\phi(X_1,\dots,X_n)$ is a discrete r.V with values in $F=\phi(E_1\times\ldots\times E_N)$ and with distribution given by

$$\forall z\in F \quad \mathbb{P}[Z=z] = \sum_{\substack{x_1\in E_1,\dots,x_n\in E_n \\ \phi(x_1,\dots,x_n)=z}} \mathbb{P}[X_1=x_1,\dots,X_n=x_n]$$

→ 7, 10

Def 2.23 Let $A\in\mathcal{F}$ be an event. We say that A occurs almost surely (a.s) if $\mathbb{P}[A]=1$ (e.g. $X\leq Y$ a.s. if $\mathbb{P}[X\leq Y]=1$)

2.4 Expectation (~average)

Def 2.25 Let $X:\Omega\to E$ be a discrete r.v. and assume $X\geq 0$ a.s. or that X is integrable (i.e. $\mathbb{E}[|X|]<\infty$). Then the expectation of X is defined by

$$\boxed{\mathbb{E}[X]=\sum_{x\in E} x\cdot\mathbb{P}[X=x]}$$

Examples:
- $X\sim \mathrm{Ber}(p)$: $\mathbb{E}[X]=p$
- $X\sim \mathrm{Poisson}(\lambda)$: $\mathbb{E}[X]=\lambda$
- Indicator of an event: $\mathbb{E}[\mathbb{1}_A]=\mathbb{P}[A]$
- $X\sim \mathrm{Geom}(p)$: $\mathbb{E}(X)=\frac{1}{p}$
- $X\sim \mathrm{Bin}(n,p)$: $\mathbb{E}(X)=n\cdot p$

Thm 2.30 (Image random variables). Let $X_1,\dots,X_n:\Omega\to E$ be n r.v., let $\phi:\mathbb{R}^n\to\mathbb{R}$. Then $Z=\phi(X_1,\dots,X_n)$ defines a discrete r.v.. If Z is integrable:

$$\boxed{\mathbb{E}[\phi(X_1,\dots,X_n)] = \sum_{x_1,\dots,x_n\in E} \phi(x_1,\dots,x_n)\,\mathbb{P}[X_1=x_1,\dots,X_n=x_n]}$$

for $n=1$:

$$\mathbb{E}[\phi(X)] = \sum_{x\in E}\phi(x)\cdot\mathbb{P}[X=x]$$

Thm 2.33 Let $X,Y:\Omega\to\mathbb{R}$ be two integrable discrete r.v., let $\lambda\in\mathbb{R}$ then $\lambda\cdot X$ and $X+Y$ are integrable discrete r.V. and:
1. $\mathbb{E}[\lambda\cdot X]=\lambda\mathbb{E}[X]$
2. $\mathbb{E}[X+Y]=\mathbb{E}[X]+\mathbb{E}[Y]$

↳ $\mathbb{E}[\lambda_1X_1+\ldots+\lambda_nX_n]=\lambda_1\mathbb{E}[X_1]+\ldots+\lambda_n\mathbb{E}[X_n]$

Thm 2.36 (Jensen's inequality) Let X be a discrete r.v. and $\phi:\mathbb{R}\to\mathbb{R}$ be a convex function.

$$\phi(\mathbb{E}[X])\leq\mathbb{E}(\phi(X))$$

↳ $|\mathbb{E}[X]|\leq\mathbb{E}[|X|]\leq\sqrt{\mathbb{E}[X^2]}$

Thm 2.37 Let X,Y be 2 discrete r.v. then the following are equivalent:

(i) X,Y are independent

(ii) $\forall a,b\in\mathbb{R}$ we have $\mathbb{P}[X=a,Y=b]=\mathbb{P}[X=a]\,\mathbb{P}[Y=b]$

(iii) $\forall f:\mathbb{R}\to\mathbb{R}, g:\mathbb{R}\to\mathbb{R}$, $\mathbb{E}[f(X)g(Y)]=\mathbb{E}[f(X)]\cdot\mathbb{E}[g(Y)]$

(same applies for n r.v. $X_1,\dots,X_n$)

falls $Y=3-X$, $\mathbb{E}[X\cdot Y]=\mathbb{E}[3X-X^2]=3\mathbb{E}[X]-\mathbb{E}[X^2]$

2.3 Variance

Def 2.39 Let X be a discrete r.v. s.t. $\mathbb{E}[X^2]<\infty$. The variance of X is defined by:

$$\sigma_X^2=\mathbb{E}[(X-\mathbb{E}[X])^2]$$

The sqrt of the variance: σ_X is the standard deviation

↳ Indicator of how large the fluctuations of X around $m=\mathbb{E}[X]$ are.

Example:
- Deterministic r.v.: $X(\omega)=a\ \forall\omega \Rightarrow \mathbb{E}[X]=a$ and $\sigma_X^2=0$
- Uniform r.v. on 2 points: Let $a<b\in\mathbb{R}$. X r.v. with $\mathbb{P}[X=a]=\mathbb{P}[X=b]=\frac{1}{2}$. Then $\mathbb{E}[X]=(a+b)/2$ and $\sigma_X=\sqrt{\mathbb{E}[(X-m)^2]}=\frac{a-b}{2}$
- Bernoulli r.v.: $\sigma_X^2=p(1-p)$
- Binomial r.v.: $\sigma_X^2=n\cdot p(1-p)$

Prop 2.41 Properties of the variance.
1. $\boxed{\sigma_X^2=\mathbb{E}[X^2]-\mathbb{E}[X]^2}$ (see 2.30 for $\mathbb{E}[X^2]$)
2. Let $X_1,\dots,X_n$ be pairwise independent r.v. and $S=X_1+\ldots+X_n$.

$$\boxed{\sigma_S^2=\sigma_{X_1}^2+\ldots+\sigma_{X_n}^2}$$

→ 9

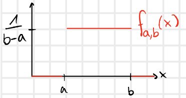
3 Continuous Random Variables

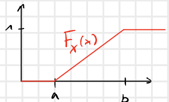
Def 1.22 A random variable $X:\Omega\to\mathbb{R}$ is said to be continuous if its distribution function F_X can be written as

$$\boxed{F_X(a)=\mathbb{P}[X\le a]=\int_{-\infty}^{a} f(x)dx \quad \forall a\in\mathbb{R}}$$

for some non-negative function $f:\mathbb{R}\to\mathbb{R}_+$, called density of X.
i.e. F_X is a continuous function

Uniform distribution $\quad X\sim\mathcal{U}([a,b])$



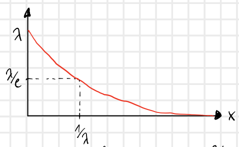


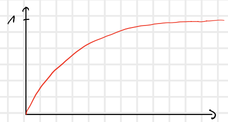
$$f_{a,b}(x)=\begin{cases}\frac{1}{b-a} & x\in[a,b]\\ 0 & x\notin[a,b]\end{cases} \qquad F_X(x)=\begin{cases}0 & x<a\\ \frac{x-a}{b-a} & a\le x\le b\\ 1 & x>b\end{cases}$$

- $\mathbb{P}[X\in[c,c+\ell]]=\frac{\ell}{b-a} \quad ([c,c+\ell]\subset[a,b])$

Exponential distribution with $\lambda>0$ $\quad T\sim Exp(\lambda)$

$$f_\lambda(x)=\begin{cases}\lambda e^{-\lambda x} & x\ge 0\\ 0 & x<0\end{cases}$$



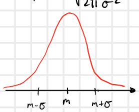


- $\mathbb{P}[T>t]=\int_t^\infty \lambda e^{-\lambda x}dx=e^{-\lambda t} \quad \forall t\ge 0$
- Absence of memory property:

$$\mathbb{P}[T>t+s\mid T>t]=\frac{\mathbb{P}[T>t+s]}{\mathbb{P}[T>t]}=\frac{e^{-\lambda(t+s)}}{e^{-\lambda t}}=e^{-\lambda s}=\mathbb{P}[T>s]$$

Normal distribution with $m\in\mathbb{R}$ and $\sigma^2>0$ $\quad X\sim\mathcal{N}(m,\sigma^2)$

$$f_{m,\sigma}(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-m)^2}{2\sigma^2}}$$



Standard Normal distribution: $X\sim\mathcal{N}(0,1)$

- If $X_1,\dots,X_n$ are independant r.v. with $(m_1,\sigma_1^2),\dots,(m_n,\sigma_n^2)$
$Z=m_0+\lambda_1X_1+\dots+\lambda_nX_n$
is a normal r.v. with $\begin{array}{l} m=m_0+\lambda_1m_1+\dots+\lambda_nm_n\\ \sigma^2=\lambda_1^2\sigma_1^2+\dots+\lambda_n^2\sigma_n^2\end{array}$
- If $X\sim\mathcal{N}(0,1)$ (i.e. X is a standard normal r.v.), then $Z=m+\sigma\cdot X$ is a normal r.v. with parameters m and σ^2. Reciprocally, $\boxed{X=\frac{Z-m}{\sigma}}$ is a standart normal.

$\mathbb{P}[X\le x]=\mathbb{P}[X<x]=\Phi(x)$ where $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{y^2}{2}}dy$
$\mathbb{P}[X\le -x]=\mathbb{P}[X\ge x]=1-\Phi(x)$
$\mathbb{P}[a\le X\le b]=\mathbb{P}[X\le b]-\mathbb{P}[X\le a]=\Phi(b)-\Phi(a)$

Example Given a table with all values of $\Phi(a)$, calculate $\mathbb{P}[0{,}43\le|Y|<1{,}54]$ where $Y\sim\mathcal{N}(-0{,}2,3)$.

$\hookrightarrow \mathbb{P}[0{,}43\le|Y|<1{,}54]=\mathbb{P}[0{,}43\le Y<1{,}54]+\mathbb{P}[0{,}43\le -Y<1{,}54]$
$=\mathbb{P}[0{,}43\le -0{,}2+3\cdot X<1{,}54]+\mathbb{P}[-0{,}43\ge -0{,}2+3X>-1{,}54]$
$=\mathbb{P}[\frac{0{,}63}{3}\le X<\frac{1{,}74}{3}]+\mathbb{P}[\frac{-0{,}23}{3}\ge X>\frac{-1{,}34}{3}]$
$\approx\mathbb{P}[0{,}21\le X<0{,}58]+\mathbb{P}[-0{,}08\ge X>-0{,}44]=\mathbb{P}[X<0{,}58]-\mathbb{P}[X\le 0{,}21]+$
$+\mathbb{P}[X\le -0{,}08]-\mathbb{P}[X<-0{,}44]=\Phi(0{,}58)-\Phi(0{,}21)+1-\Phi(0{,}08)-1+\Phi(0{,}44)$
$=\Phi(0{,}58)-\Phi(0{,}21)+\Phi(0{,}44)-\Phi(0{,}08)$

- The "probability mass" is mainly in the interval $[m-3\sigma,m+3\sigma]$.
Namely, we have $\mathbb{P}[|X-m|\ge 3\sigma]\le 0{,}0027$

Cauchy distribution

$$f(x)=\frac{1}{\pi}\frac{1}{1+x^2}$$

Thm 3.2 (How to recognize a continuous r.v.)
Let X be a r.v.. Assume F_X is continuous and piecewise C^1 on every interval (x_i,x_{i+1}), i.e. $\exists\, x_0=-\infty<x_1<\dots<x_{n-1}<x_n=+\infty$ s.t. F_X is C^1 on every interval (x_i,x_{i+1}). Then X is a continuous r.v. and a density f can be constructed by defining
$$f(x)=F_X'(x) \quad \forall x\in(x_i,x_{i+1})$$
and setting arbitrary values at $x_1,\dots,x_{n-1}$

Question: Let X be a r.v. with density f. Assuming it exists, what is the density of $Y=\phi(X)$?
1/ Compute the distribution function: $F_Y(x)=\mathbb{P}[\phi(X)\le x]=\dots$
2/ If F_Y is continuous and piecewise C^1, then $f(x)=F_Y'(x)$ on each interval where the derivative is defined

⚠ Be careful to indicate the domains where $f(x)=0$ (e.g. $x<0,\dots$)

3.2 Expectation

Def 3.5 Let $X:\Omega\to\mathbb{R}$ be a continuous r.v. with density f. We say that X is integrable if $\mathbb{E}[|X|]<\infty$. Then the expectation of X is

$$\boxed{\mathbb{E}[X]=\int_{-\infty}^{\infty}x\,f(x)dx}$$

If $X\ge 0$ a.s. then $f(x)=0$ for $x<0$ and
$\mathbb{E}[X]=\int_0^\infty x\,f(x)dx$

Def 3.6 Let X be a r.v. with density f. Let $\phi:\mathbb{R}\to\mathbb{R}$ be such that
$$\int_{-\infty}^{\infty}|\phi(x)|f(x)dx<\infty$$
Then we define the expectation of $\phi(X)$ by
$$\boxed{\mathbb{E}[\phi(X)]=\int_{-\infty}^{\infty}\phi(x)f(x)dx} \overset{!}{=} \underbrace{\int_{-\infty}^{\infty}x\,g(x)dx}_{\text{density of }\phi(X)}$$

Examples:
- $X\sim\mathcal{U}([a,b])$
$\mathbb{E}[X]=\frac{1}{b-a}\int_a^b x\,dx=\frac{1}{b-a}(\frac{1}{2}b^2-\frac{1}{2}a^2)=\frac{a+b}{2}$
- $X\sim Exp(\lambda)$
$\mathbb{E}[X]=\int_0^\infty x\lambda e^{-\lambda x}dx=[-xe^{-\lambda x}]_0^\infty+\int_0^\infty e^{-\lambda x}dx=\frac{1}{\lambda}$
- $X\sim\mathcal{N}(m,\sigma^2)$
First we see that $X=\phi(Y)=m+\sigma\cdot Y$, where $Y\sim\mathcal{N}(0,1)$
$\mathbb{E}[X]=\mathbb{E}[m+\sigma\cdot Y]=m+\sigma\,\mathbb{E}[Y]=m+\int_{-\infty}^{\infty}x\,f_{0,1}(x)dx=m$
- $X\sim$ Cauchy, the expectation is not well defined!
$\mathbb{E}[|X|]=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|x|}{1+x^2}dx=+\infty$

Thm 3.8 (Jensen's inequality). Let $\phi:\mathbb{R}\to\mathbb{R}$ be a convex function. Let X be a continuous r.v. s.t. $\mathbb{E}[X]$ and $\mathbb{E}[\phi(X)]$ are well defined, then
$$\phi(\mathbb{E}[X])\le\mathbb{E}[\phi(X)]$$

3.3 Variance

Def 3.9 Let $X:\Omega\to\mathbb{R}$ be a continuous r.v. with density f. If $\mathbb{E}[X^2]<\infty$, then we define the variance of X by

$$\sigma_X^2=\mathbb{E}[(X-m)^2]=\int_{-\infty}^{\infty}(x-m)^2\cdot f(x)\,dx$$

where $m=\mathbb{E}[X]$

Prop 3.11 Properties of the Variance

1. Let X be a continuous r.v. with $\mathbb{E}[X^2]<\infty$. Then $\boxed{\sigma_X^2=\mathbb{E}[X^2]-\mathbb{E}[X]^2}$
2. Let X be a continuous r.v. with $\mathbb{E}[X^2]<\infty$, let $\lambda,\mu\in\mathbb{R}$. Then $\sigma^2_{\lambda X+\mu}=\lambda^2\sigma_X^2$
3. Let $X_1,\dots,X_n$ be n pairwise independent continuous r.v. with $\mathbb{E}[X_i^2]<\infty$, let $S=\lambda_1X_1+\dots+\lambda_nX_n$. Then $\boxed{\sigma_S^2=\lambda_1^2\sigma_{X_1}^2+\dots+\lambda_n^2\sigma_{X_n}^2}$

Examples:
- $X\sim U([a,b])$ then $X=a+(b-a)Y$ where $Y\sim([0,1])$

$$\sigma_Y^2=\int_0^1 x^2\,dx-m^2=\frac{1}{3}-\left(\frac{1}{2}\right)^2=\frac{1}{12}$$

Then $\sigma_X^2=(b-a)^2\cdot\sigma_Y^2=\frac{1}{12}(b-a)^2$

- $X\sim Exp(\lambda)$

$$\mathbb{E}[X^2]=\int_0^\infty x^2\lambda e^{-\lambda x}dx=\left[-x^2e^{-\lambda x}\right]_0^\infty+2\int_0^\infty xe^{-\lambda x}dx=0+\frac{2}{\lambda}\mathbb{E}[X]=\frac{2}{\lambda^2}$$

$$\sigma_X^2=\mathbb{E}[X^2]-\mathbb{E}[X]^2=\frac{2}{\lambda^2}-\frac{1}{\lambda^2}=\frac{1}{\lambda^2}$$

- $X\sim\mathcal{N}(m,\sigma^2)$ $\quad X=m+\sigma Y$ where $Y\sim\mathcal{N}(0,1)$

$$\sigma_Y^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}x^2e^{-x^2/2}dx=\frac{1}{2\pi}\left[-x\cdot e^{-x^2/2}\right]_{-\infty}^{\infty}+\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{x^2/2}dx=0+1=1$$

$$\sigma_X^2=\sigma^2\cdot\sigma_Y^2=\sigma^2$$

SO19 A3.b):

Seien $X\sim Exp(1)$, $Y\sim U([1,3])$ unabhängig, sei $u>0$.
Berechnen Sie erst $\boxed{\mathbb{P}[X>uY]}$ und verwenden Sie dann dieses Resultat, um die Verteilungsfunktion F_u von $U=X/Y$ zu bestimmen.

↳ $f_X(x)=e^{-x}\cdot\mathbb{1}_{\{x\ge0\}}$, $f_Y(y)=\frac{1}{3-1}\cdot\mathbb{1}_{\{1\le y\le3\}}$

$$\mathbb{P}[X>uY]=\iint_{-\infty}^{\infty}\mathbb{1}_{\{x>uy\}}f_X(x)f_Y(y)\,dx\,dy=\frac{1}{2}\int_1^3\int_0^\infty\mathbb{1}_{\{x>u\cdot y\}}e^{-x}\,dx\,dy$$

$$=\frac{1}{2}\int_1^3\int_{u\cdot y}^{\infty}e^{-x}\,dx\,dy=\frac{1}{2}\int_1^3\left[-e^{-x}\right]_{uy}^{\infty}dy=\frac{1}{2}\int_1^3e^{-uy}dy=\frac{1}{2}\left[-\frac{1}{u}e^{-uy}\right]_1^3=$$

$$=\frac{1}{2u}\left(e^{-u}-e^{-3u}\right)$$

$$F_u=\mathbb{P}[U\le u]=\mathbb{P}\left[\frac{X}{Y}\le u\right]=\mathbb{P}[X\le u\cdot Y]=1-\mathbb{P}[X>uY]=1-\frac{1}{2u}e^{-u}\left(1-e^{-2u}\right)$$

3.4 Joint distributions

Def 3.12 Two random variables $X,Y:\Omega\to\mathbb{R}$ have a continuous joint distribution if there exists a function $f:\mathbb{R}^2\to\mathbb{R}_+$ s.t.

$$\mathbb{P}[X\in[a,a'],Y\in[b,b']]=\int_a^{a'}\left(\int_b^{b'}f(x,y)\,dy\right)dx$$

$\forall\ -\infty\le a\le a'\le\infty$ and $-\infty\le b\le b'\le\infty$. $f(x,y)$ is called joint density of (X,Y)

- By choosing $a=b=-\infty$ and $a'=b'=\infty$

$$\mathbb{P}[X\in[-\infty,\infty],Y\in[-\infty,\infty]]=1$$

- $\boxed{\mathbb{E}[\phi(X,Y)]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x,y)\cdot f_{X,Y}(x,y)\,dx\,dy}$

Thm 3.14 Let X,Y be two continuous r.v. with densities f_X and f_Y.
The following are equivalent
(i) X,Y are independent
(ii) X,Y are jointly continuous with joint density $f_{X,Y}(x,y)=f_X(x)\cdot f_Y(y)$
i.e. $\mathbb{P}[X\in[a,a'],Y\in[b,b']]=\mathbb{P}[X\in[a,a']]\cdot\mathbb{P}[Y\in[b,b']]$
(iii) For every $\phi:\mathbb{R}\to\mathbb{R}$, $\psi:\mathbb{R}\to\mathbb{R}$,

$$\mathbb{E}[\phi(X)\psi(Y)]=\mathbb{E}[\phi(X)]\mathbb{E}[\psi(Y)]$$

↳ 2 independent continous r.v. are automatically jointly continuous.

Marginal densities

If X,Y possess a joint density $f_{X,Y}$, then we have

$$\mathbb{P}[X\le a]=\mathbb{P}[X\in[-\infty,a],Y\in[-\infty,\infty]]=\int_{-\infty}^{a}\left(\int_{-\infty}^{\infty}f_{X,Y}(x,y)\,dy\right)dx$$

And therefore: $f_X(x)=\int_{-\infty}^{\infty}f_{X,Y}(x,y)\,dy$

$f_Y(y)=\int_{-\infty}^{\infty}f_{X,Y}(x,y)\,dx$

Examples: 60-61, 63

- $X\sim\mathcal{N}(\mu_x,\sigma_x)$, $Y\sim\mathcal{N}(\mu_y,\sigma_y)$ independent $\Rightarrow X+Y\sim\mathcal{N}\left(\mu_x+\mu_y,\sqrt{\sigma_x^2+\sigma_y^2}\right)$

Example

$$f_{R,A}(r,a)=\begin{cases}4are^{-r^2} & \text{falls } r\ge0 \text{ und } 0\le a\le r\\ 0 & \text{sonst}\end{cases}$$

Bestimmen Sie die Randdichte von R und A:

$$f_R(r)=\int_{-\infty}^{\infty}f_{R,A}(r,a)\,da=\int_{-\infty}^{\infty}4are^{-r^2}\mathbb{1}_{\{r\ge0\}}\mathbb{1}_{\{0\le a\le r\}}da=\int_0^r4are^{-r^2}\mathbb{1}_{\{r\ge0\}}da=$$

$$=4re^{-r^2}\frac{r^2}{2}\mathbb{1}_{\{r\ge0\}}=2r^3e^{-r^2}\mathbb{1}_{\{r\ge0\}}$$

$$f_A(r)=\int_{-\infty}^{\infty}f_{R,A}(r,a)\,da=\int_{-\infty}^{\infty}4are^{-r^2}\mathbb{1}_{\{r\ge0\}}\mathbb{1}_{\{0\le a\le r\}}dr=4a\,\mathbb{1}_{\{a\ge0\}}\int_a^\infty re^{-r^2}dr=2ae^{-a^2}\mathbb{1}_{\{a\ge0\}}$$

4 Asymptotic results

Thm 3.16: (law of large numbers) Assume $\mathbb{E}[|X_1|]$ is well defined and finite. Defining $m=\mathbb{E}[X_1]$, we have

$$\boxed{\lim_{n\to\infty}\frac{X_1+\dots+X_n}{n}=m\ \text{a.s.}}$$

Example:
- If $X_1,X_2,\dots$ is an infinite sequence of i.i.d. Bernoulli r.v. with parameters p. Then $\lim_{n\to\infty}\frac{X_1+\dots+X_n}{n}=p$ a.s.
- If $T_1,T_2,\dots$ is an infinite sequence of i.i.d. Exponential r.v. with parameters λ. Then $\lim_{n\to\infty}\frac{T_1+\dots+T_n}{n}=\lambda$ a.s.

↳ The law of large number tells us that for large n, the empirical avrg. is close to the expectation $m=\mathbb{E}[X_1]$. We might ask ourself: how far is $\frac{X_1+\dots+X_n}{n}$ from m typically?
If we consider iid normal distribution with expectation m an variance σ^2 then the r.v. $\frac{X_1+\dots+X_n-n\cdot m}{\sqrt{\sigma^2n}}$ is a standard normal.

Thm 3.18: (Central limit Theorem) Assume that $\mathbb{E}[X_1^2]$ is well defined and finite. Defining $m=\mathbb{E}[X_1]$ and $\sigma^2=Var(X_1)$

$$\mathbb{P}\left[\frac{S_n-n\cdot m}{\sqrt{\sigma^2n}}\le a\right]\xrightarrow[n\to\infty]{}\Phi(a)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a}e^{-\frac{x^2}{2}}dx$$

Lookup-sheet

Das Urnenmodell

Variation mit Wiederholung: $\underbrace{n \cdot n \cdot \dots \cdot n}_{k} = n^k$

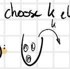
Variation ohne Wiederholung: $\underbrace{n \cdot n-1 \cdot \dots \cdot n-k+1}_{k} = \frac{n!}{(n-k)!}$

Kombination ohne Wiederholung:

Für $k \leq n$: $n \cdot (n-1) \cdot \dots \cdot (n-k+1) / \underbrace{k!}_{\#\text{Permutationen}} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k}$

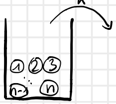
choose k elements from n total elements / don't care in what order

Kombination mit Wiederholung:



der Möglichkeiten k von n Elem auszuwählten wobei Mehrfachauswahl möglich $\Big\} = \underbrace{n \cdot n \cdot n \cdot n \cdot \dots \cdot n}_{k \text{ mal}} / \underbrace{k!}_{\#\text{Permutationen}} = \frac{n^k}{k!} = \frac{(n+k-1)!}{(n-1)!\,k!} = \binom{n+k-1}{k}$

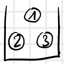
$n \cdot (n-1) \cdot (n-2)$



k

	mit zurücklegen	ohne zurücklegen (gilt nur für $k \leq n$)
Reihenfolge relevant (Variation) → bilden Folgen	a) n^k	b) $\frac{n!}{(n-k)!}$
Reihenfolge irrelevant (Kombination) → bilden Mengen	c) $\binom{n+k-1}{k}$	d) $\frac{n!}{(n-k)!} \cdot \frac{1}{k!} = \binom{n}{k}$

Bsp: •



12
21
31
13
23
32
6
11
22
33
9

{1,2} {1,1}
{1,3} {2,2}
{2,3} {3,3}
3 6

Lotto Spielen: N Elemente, R richtige, ziehe m ohne zurücklegen, davon k richtige, m-k falsch

3 Ziehungen, 2 blau zu ziehen:

Mögliche Ziehungen mit k richtige: $\binom{R}{k} \cdot \binom{N-R}{m-k}$

Mögliche Ziehungen insgesamt: $\binom{N}{m}$

$\Rightarrow$ Wahrscheinlichkeit k richtige zu ziehen: $\frac{\binom{R}{k} \cdot \binom{N-R}{m-k}}{\binom{N}{m}}$

Identitäten

- $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$
- $\sum_{j=0}^{k} \binom{m}{k-j}\binom{n}{j} = \binom{m+n}{k}$
- $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $\int_0^\infty t^n e^{-t}\,dt = n!$

$dx/dt = \lambda$
$x = \lambda t$

⚠ $\int_0^\infty (\lambda t)^n e^{-\lambda t}\,dt \overset{\text{subst}}{=} \int_0^\infty x^n e^{-x} \frac{dx}{\lambda} = \frac{n!}{\lambda}$

- $\int_a^b u(x) \cdot \frac{d\,v(x)}{dx}\,dx = \left[u(x)v(x)\right]_a^b - \int_a^b \frac{du(x)}{dx} \cdot v(x)\,dx$
- $\frac{d}{dx} \ln(x) = \frac{1}{x}$

0 1 2 ...

0 →	1
1 →	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1
6	1 6 15 20 15 6 1
7	1 7 21 35 35 21 7 1
8	1 8 28 56 70 56 28 8 1
9	1 9 36 84 126 126 84 36 9 1
10	1 10 45 120 210 252 210 120 45 10 1

$\binom{5}{3} = 10$

$\binom{8}{6} = 28$

Zusätzliches

1/ 2-Fache Würfelwurf: $\Omega=\{1,\dots,6\}^2$, $\mathcal{F}=\mathcal{P}(\Omega)$, Laplace-Wahrscheinlichkeitsmass $\mathbb{P}$.

Betrachte die Abbildung $X:\Omega\to\mathbb{R}$ gegeben durch $X(\omega)=|\omega_1-\omega_2|$.

- Wieso ist X ein Zufallsvariable?

Weil $\forall a\in\mathbb{R}$ $\{\omega\in\Omega: X(\omega)\le a\}\in\mathcal{F}$ gilt da $\mathcal{F}=\mathcal{P}(\Omega)$

- Berechne die Verteilungsfunktion F_X von X:

$$\mathbb{P}[X=a]=\begin{cases} 0 & a<0 \text{ oder } a>5\\ 1/6 & a=0\\ \left(\frac{1}{6}\cdot 2+\frac{2}{6}\cdot 4\right)\frac{1}{6}=\frac{5}{18} & a=1\\ \left(\frac{1}{6}\cdot 4+\frac{2}{6}\cdot 2\right)\frac{1}{6}=\frac{4}{18}=\frac{2}{9} & a=2\\ 1/6 & a=3\\ \left(\frac{1}{6}\cdot 4+\frac{0}{6}\cdot 2\right)\frac{1}{6}=\frac{4}{36}=\frac{1}{9} & a=4\\ \left(\frac{1}{6}\cdot 2+\frac{0}{6}\cdot 4\right)\frac{1}{6}=\frac{2}{36}=\frac{1}{18} & a=5\end{cases}$$

$$F_X(x)=\mathbb{P}[X\le x]=\begin{cases} 0 & x<0\\ \frac{1}{6} & 0\le x<1\\ \frac{1}{6}+\frac{5}{18}=\frac{4}{9} & 1\le x<2\\ \frac{4}{9}+\frac{2}{9}=\frac{2}{3} & 2\le x<3\\ \frac{2}{3}+\frac{1}{6}=\frac{5}{6} & 3\le x<4\\ \frac{5}{6}+\frac{1}{9}=\frac{17}{18} & 4\le x<5\\ \frac{17}{18}+\frac{1}{18}=1 & x\ge 5\end{cases}$$

2/ Gleichverteilung auf [0,1]

$\Omega=[0,1]^2$, $\mathcal{F}$ Borel-σ-Algebra auf $[0,1]$: kleinste Menge von Teilmengen von Ω, die H1–H3 erfüllt und alle Mengen $[x_1,x_2]\times[y_1,y_2]$, $0\le x_1\le x_2\le 1$, $0\le y_1\le y_2\le 1$ enthält.

$Y:\Omega\to\mathbb{R}$ $\quad Y(\omega)=2\omega_1+2\omega_2$

- Zeige, dass Y ein Z.V. ist:

für $a<0$: $\{\omega\in\Omega: Y(\omega)\le a\}=\{\omega\in[0,1]^2: 2\omega_1+2\omega_2\le a\}=\emptyset\in\mathcal{F}$

für $a\ge 4$: $\{\omega\in\Omega: Y(\omega)\le a\}=\{\omega\in[0,1]: 2\omega_1+2\omega_2\le a\}=\Omega\in\mathcal{F}$

für $a\in[0,4)$: $\{\omega\in\Omega: Y(\omega)\le a\}=\{\omega\in[0,1]^2: 2\omega_1+2\omega_2\le a\}\in\mathcal{F}$

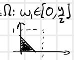
- Berechne F_Y von Y:

für $y<0$: $F_Y(y)=\mathbb{P}[Y\le y]=\mathbb{P}[\emptyset]=0$

für $y\ge 4$: $F_Y(y)=\mathbb{P}[Y\le y]=\mathbb{P}[\Omega]=1$

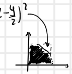
für $y\in[0,2)$: $F_Y(y)=\mathbb{P}[Y\le y]=\mathbb{P}[2\omega_1+2\omega_2\le y]=\mathbb{P}[\{\omega\in\Omega: \omega_1\in[0,\frac{y}{2}],\omega_2\in[0,\frac{y}{2}-\omega_1]\}]$

$=\text{Area}(\{\omega\in\Omega: 2\omega_1+2\omega_2\le y\})=\frac{y^2}{8}$



für $y\in[2,4)$: $1-F_Y(y)=\mathbb{P}[Y>y]=\text{Area}(\{\omega\in\Omega: 2\omega_1+2\omega_2>y\})=\frac{1}{2}(2-\frac{y}{2})^2$

$F_Y(y)=1-\frac{1}{2}(2-\frac{y}{2})^2=y-\frac{y^2}{8}-1$



3/ 5 faire Münzen

… werden nacheinander geworfen. $\Omega=\{\overset{\text{Zahl}}{0},\overset{\text{Kopf}}{1}\}^5$, $\mathcal{F}=\mathcal{P}(\Omega)$, $\mathbb{P}=\frac{|A|}{|\Omega|}=\frac{|A|}{2^5}$

$X\,\hat{=}$ total Anzahl erschienenen Köpfe $:=\{\omega\in\Omega: \sum\omega_i\}$

$$F_X(a)=\mathbb{P}[X\le a]=\sum_{i\le x,\,0\le i\le 5}\frac{\binom{5}{i}}{32}=\begin{cases} 0 & a<0\\ 1/2^5= \frac{1}{32} & 0\le a<1\\ \frac{1}{32}+\frac{5}{2^5}=\frac{6}{32}=\frac{3}{16} & 1\le a<2\\ \frac{6}{32}+\frac{\binom{5}{2}}{2^5}=\frac{6+\frac{5\cdot 4}{2}}{32}=\frac{16}{32}=\frac{1}{2} & 2\le a<3\\ \frac{1}{2}+\binom{5}{3}/2^5=\frac{16+10}{32}=\frac{26}{32}=\frac{13}{16} & 3\le a<4\\ \frac{13}{16}+\binom{5}{4}/2^5=\frac{26+5}{32}=\frac{31}{32} & 4\le a<5\\ 1 & a\ge 5\end{cases}$$

Wir setzen 1 Franken Einsatz bei jedem Wurf +10% des Einsatz wenn Kopf, −10% wenn Zahl erscheint. $Y\,\hat{=}$ totalen Geldbetrag nach fünf Würfe.

$$\Rightarrow Y=(1.1)^X\cdot(0,9)^{5-X}\Rightarrow F_Y(y)=\begin{cases} 0 & y<0,9^5\\ 1/32 & 0,9^5\le y<1,1\cdot 0,9^4\\ 3/16 & 1,1\cdot 0,9^4\le y<1,1^2 0,9^3\\ 1/2 & 1,1^2 0,9^3\le y<1,1^3 0,9^2\\ 13/16 & 0,9^2 1,1^3\le y<1,1^4\cdot 0,9\\ 31/32 & 0,9\cdot 1,1^4\le y<1,1^5\\ 1 & \text{sonst}\end{cases}$$

4/ Indikatorfunktion

Wir betrachten einen Wahrscheinlichkeitsraum $(\Omega,\mathcal{F},\mathbb{P})$ und $A\in\mathcal{F}$. Indikatorfunktion ist definiert durch:

$$\mathbb{1}_A(\omega)=\begin{cases}0 & \text{if } \omega\notin A\\ 1 & \text{if } \omega\in A\end{cases}\qquad\forall\omega\in\Omega$$

→ Zeigen $\mathbb{1}_A$ ist ein Zufallsvariable.

$$\{\mathbb{1}_A\le a\}=\{\omega\in\Omega: \mathbb{1}_A(\omega)\le a\}=\left.\begin{cases}\emptyset & a<0\\ A^c & 0\le a<1\\ \Omega & a\le 1\end{cases}\right\}\in\mathcal{F}\qquad\forall a\in\mathbb{R}$$

Also $\emptyset, A^c, \Omega\in\mathcal{F}$

5/ Bedingte Wahrscheinlichkeit

Sei $Y\sim\text{Poisson}(\lambda)$ unabhängig von X. Ist die Ausdruck $=\mathbb{P}[X=n]=\frac{\lambda^n}{n!(e^\lambda-1)}$?

(i) $\mathbb{P}[X=Y\mid Y=n]=\frac{\mathbb{P}[(X=Y)\cap(Y=n)]}{\mathbb{P}[Y=n]}=\frac{\mathbb{P}[(X=n)\cap(Y=n)]}{\mathbb{P}[Y=n]}\underset{\text{unabhängig}}{=}\frac{\mathbb{P}[X=n]\cdot\cancel{\mathbb{P}[Y=n]}}{\cancel{\mathbb{P}[Y=n]}}$ ✓

(ii) $\mathbb{P}[Y=n\mid Y\neq 0]=\frac{\mathbb{P}[Y=n\cap Y\neq 0]}{\mathbb{P}[Y\neq 0]}=\frac{\mathbb{P}[Y=n]}{1-\mathbb{P}[Y=0]}=\frac{\lambda^n}{n!}e^{-\lambda}\cdot\frac{1}{1-e^{-\lambda}}=\frac{\lambda^2}{n!(e^\lambda-1)}$ ✓

(iii) $\mathbb{P}[Y\neq 0\mid Y=n]=\frac{\mathbb{P}[Y\neq 0\cap Y=n]}{\mathbb{P}[Y=n]}=\begin{cases}0 & n=0\\ 1 & \text{sonst}\end{cases}$ ×

(iv) $\mathbb{P}[X=n\mid Y=0]=\frac{\mathbb{P}[X=n\cap Y=0]}{\mathbb{P}[Y=0]}=\frac{\mathbb{P}[X=n]\,\cancel{\mathbb{P}[Y=0]}}{\cancel{\mathbb{P}[Y=0]}}=\mathbb{P}[X=n]$ da unabhängig ✓

6/ Pokerspiel

1 Gegner, 52 Karten (4 Farben, 13 verschiedene Kartenwerte), jeder Spieler bekommt 5 Karten:

(i) Angenommen wir wissen der Gegner mind. eine Dame in der Hand hat. Wie hoch ist die Wahrscheinlichkeit, dass er mind. 2 Dame in der Hand hat?

↳ Sei A_i $(i=0,1,2,3,4)$ das Ereignis, dass der Gegner genau i Damen hat. Diese sind paarweise disjunkt. $\mathbb{P}(A_i)=\frac{\binom{4}{i}\binom{52-4}{5-i}}{\binom{52}{5}}=\frac{|A_i|}{|\Omega|}$

$$\mathbb{P}(\overbrace{A_2\cup A_3\cup A_4}^{\text{mind. 2 Dame}}\mid\overbrace{A_1\cup A_2\cup A_3\cup A_4}^{\text{mind. 1 Dame}})=\frac{\mathbb{P}(A_2\cup A_3\cup A_4)}{\mathbb{P}(A_1\cup A_2\cup A_3\cup A_4)}=\frac{1-\mathbb{P}(A_0)-\mathbb{P}(A_1)}{1-\mathbb{P}(A_0)}=\frac{\binom{52}{5}-\binom{48}{5}-\binom{4}{1}\binom{48}{4}}{\binom{52}{5}-\binom{48}{5}}\ (\approx 0{,}12)$$

(ii) Jetzt nehmen wir an, wir wissen er hat die Herz Dame. Wie hoch ist die Wahrscheinlichkeit, dass er mind. 2 Damen hat?

$B:=$ Ereignis, dass der Gegner die Herz Dame hat: $\mathbb{P}[B]=\frac{\binom{1}{1}\binom{52-1}{4}}{\binom{52}{5}}$

$A_i:=$ Ereigniss, dass der Gegner mind. i Damen hat.

$$\mathbb{P}(A_i\cap B)=\frac{\underbrace{\binom{1}{1}}_{\text{Herz Dame}}\overbrace{\binom{3}{i-1}}^{\text{andere Damen}}\overbrace{\binom{52-4}{5-i}}^{\text{Restliche Karte}}}{\binom{52}{5}}$$

$$\mathbb{P}[A_2\cup A_3\cup A_4\mid B]=\frac{\mathbb{P}((A_2\cup A_3\cup A_4)\cap B)}{\mathbb{P}(B)}=\frac{\mathbb{P}[(A_2\cap B)\cup(A_3\cap B)\cup(A_4\cap B)]}{\mathbb{P}[B]}=\frac{\mathbb{P}[A_2\cap B]+\mathbb{P}[A_3\cap B]+\mathbb{P}[A_4\cap B]}{\mathbb{P}[B]}$$

7/ Defekter Stellen

Die Anzahl Y defekter Stellen auf einem Chip sei $\sim\text{Poisson}(\lambda)$. Sei X die Anzahl der Fehler auf einem bestimmten Teilgebiet T des Chips. Wir nehmen an, dass sich jeder der Fehler unabhängig von den anderen mit Wahrscheinlichkeit $p\in(0,1)$ in diesem Teilgebiet befindet.

a) Bestimme die Verteilung von X.

Die Annahme bedeutet $\mathbb{P}[X=j\mid Y=k]=\binom{k}{j}p^j(1-p)^{k-j}$,

Quasi: gegeben $Y=k$, Fehler $1\in T$ mit Wahrsch. p, …, Fehler $k\in T$ mit Wahrsch. p

Also $\sim\text{Bin}(k,p)$

$$\mathbb{P}[X=j]\overset{\text{totale Wahrscheinlichkeit}}{=}\sum_{k\ge j}\mathbb{P}[X=j,Y=k]\overset{\text{bedingte Wahrsch.}}{=}\sum_{k\ge j}\mathbb{P}[X=j\mid Y=k]\,\mathbb{P}[Y=k]$$

$$=\sum_{k\ge j}\binom{k}{j}p^j(1-p)^{k-j}\cdot\frac{\lambda^k}{k!}e^{-\lambda}\overset{k'=k-j}{=}\sum_{k\ge 0}\binom{k+j}{j}p^j(1-p)^k\cdot\frac{\lambda^{k+j}}{(k+j)!}e^{-\lambda}$$

$$=p^je^{-\lambda}\sum_{k\ge 0}\frac{\cancel{(k+j)!}}{k!\,j!}\frac{\lambda^{k+j}}{\cancel{(k+j)!}}(1-p)^k=\frac{p^j\lambda^j}{j!}e^{-\lambda}\sum_{k\ge 0}\frac{(\lambda(1-p))^k}{k!}$$

$$=\frac{(p\lambda)^je^{-\lambda}}{j!}e^{\lambda(1-p)}=\frac{(p\lambda)^je^{\lambda p}}{j!}\Rightarrow X\sim\text{Poisson}(p\lambda)$$

8/ Nachrichtenkanal

... überträgt binäre Codewörter zu je 1024 Bits. Ein Bit wird falsch übertragen mit W. p.

$X := \#$ falsch übertragener Bits in einem Codewort.

$\mathbb{P}[X=j] = \binom{1024}{j} p^j (1-p)^{1024-j}$ für $0 \le j \le 1024$

9/

Sei X r.v., $\lambda > 0$ $\mathbb{P}[X=n] = c\frac{\lambda^n}{n!}$ für $n \in \mathbb{N}$

a) Bestimmen sie die $\in c$:

$$\sum_{n>0} \mathbb{P}[X=n] \overset{!}{=} 1 \Rightarrow \sum_{n=1}^{\infty} c\frac{\lambda^n}{n!} = c\left(\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} - 1\right) = c(e^\lambda - 1) \overset{!}{=} 1$$

$$c = \frac{1}{e^\lambda - 1}$$

b) Bestimmen sie Erwartungswert und Varianz von X.

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} n \cdot c\frac{\lambda^n}{n!} = c\lambda \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{(n-1)!} = c\lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = c\lambda e^\lambda = \frac{\lambda e^\lambda}{e^\lambda - 1}$$

$$\mathbb{E}[X^2] = \sum_{n=1}^{\infty} n^2 \cdot c\frac{\lambda^n}{n!} = \sum_{n=1} n\, c\frac{\lambda^n}{n!} + \sum_{n=1} n(n-1)\, c\frac{\lambda^n}{n!} \overset{\text{für } n=1,\ n(n-1)=0}{=}$$

$$= \mathbb{E}[X] + c\lambda^2 \sum_{n=2}^{\infty} \frac{\lambda^{n-2}}{(n-2)!} = \mathbb{E}[X] + c\lambda^2 e^\lambda = c(\lambda e^\lambda + \lambda^2 e^\lambda)$$

$$\sigma_X^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{\lambda e^\lambda + \lambda^2 e^\lambda}{e^\lambda - 1} - \frac{\lambda^2 e^{2\lambda}}{(e^\lambda - 1)^2} = \frac{e^\lambda \lambda (e^\lambda - \lambda - 1)}{(e^\lambda - 1)^2}$$

9/ Indikatorfunktion / Zufallsvariable

Wir betrachten einen Wahrscheinlichkeitsraum $(\Omega, \mathcal{F}, P)$ und $A \in \mathcal{F}$.

$\forall \omega \in \Omega \quad \mathbb{1}_A(\omega) = \begin{cases} 0 & \text{if } \omega \notin A \\ 1 & \text{if } \omega \in A \end{cases}$

a) Zeigen Sie, dass $\mathbb{1}_A$ eine Zufallsvariable ist, und Berechnen Sie ihre Verteilungsfunktion.

↳ Wir müssen nachweisen, dass $\forall a \in \mathbb{R}$ gilt $\{\omega \in \Omega : \mathbb{1}(\omega) \le a\} \in \mathcal{F}$

$$\{\mathbb{1}_A \le a\} = \{\omega \in \Omega : \mathbb{1}_A(\omega) \le a\} = \begin{cases} \emptyset & a < 0 \\ A^c & 0 \le a < 1 \\ \Omega & a \ge 1 \end{cases} \text{, jeweils } \in \mathcal{F}$$

$$F_{\mathbb{1}_A}(a) = \mathbb{P}[\mathbb{1}_A \le a] = \begin{cases} 0 & a < 0 \\ \mathbb{P}[A^c] = 1 - \mathbb{P}[A] & 0 \le a < 1 \\ 1 & a \ge 1 \end{cases}$$

10/ Gemeinsame Verteilung auf eigene Verteilung bei diskrete r.v.

$X_1, X_2: \Omega \to \{1,2,3,4,5\}$. $\mathbb{P}[X_1 = i, X_2 = j] = \begin{cases} p & i \ne j \\ \frac{1}{5} - 4p & i = j \end{cases}$. Berechne Verteilung von X_1, X_2

$$\mathbb{P}[X_1 = i] = \sum_{j=1}^{5} \mathbb{P}[X_1 = i \cap X_2 = j] = 4p + \frac{1}{5} - 4p = \frac{1}{5} \quad \forall i \in \{1,\dots,5\}$$

$$\mathbb{P}[X_2 = j] \overset{\text{wegen Symmetrie}}{=} \mathbb{P}[X_1 = i] = \frac{1}{5} \quad \forall i \in \{1,\dots,5\}$$

11/ Urnenmodell

RRRR
BB

$X = \#B$

$Y = \#Y$

3 Ziehungen **mit** Zurücklegen: Binomial Verteilt!

$$\mathbb{P}[X=j] = \underbrace{\left(\frac{2}{6}\right)^j \left(\frac{4}{6}\right)^{3-j}}_{\text{Wahrscheinlichkeit so ein Ziehung zu machen}} \binom{3}{j}$$

↖ # Möglichkeiten Ziehung mit j B und $3-j$ R zu ordnen. (choose j places out of 3)

3 Ziehungen **ohne** Zurücklegen:

$$\mathbb{P}[X=j] = \begin{cases} 0 & j=3 \\ \frac{2}{6} \cdot \frac{1}{5} \cdot \frac{4}{4} + \frac{4}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} + \frac{2}{6} \cdot \frac{4}{5} \cdot \frac{1}{4} = \frac{3}{15} = \frac{1}{5} & j=2 \\ \frac{2}{6} \cdot \frac{4}{5} \cdot \frac{3}{4} + \frac{4}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} + \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{5} & j=1 \\ \frac{4}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} = \frac{1}{5} & j=0 \end{cases}$$

Oder Lotto: (ergibt die gleiche Resultate)

$$\mathbb{P}[X=j] = \frac{\binom{2}{j}\binom{4}{3-j}}{\binom{6}{3}}.$$

12/ Inklusions-Exklusionsprinzip Beweis über Induktion

Verankerung $n=1$: $\mathbb{P}\left(\bigcup_{i=1}^{1} A_i\right) = A_1$ ✓

$$n=2: \mathbb{P}\left(\bigcup_{i=1}^{2} A_i\right) = \sum_{k=1}^{2} (-1)^{k-1} \sum_{1 \le i_1 < \dots < i_k \le 2} \mathbb{P}[A_{i_1} \cap \dots \cap A_{i_k}]$$

$$= \sum_{1 \le i_1 \le 2} \mathbb{P}[A_{i_1}] - \sum_{1 \le i_1 < i_2 \le 2} \mathbb{P}[A_{i_1} \cap A_{i_2}] \checkmark$$

Induktionsschritt: nehmen an gleichung gilt.

$$\mathbb{P}\left[\bigcup_{i=1}^{n+1} A_i\right] = \mathbb{P}\left[\bigcup_{i=1}^{n} A_i \cup A_{n+1}\right] \overset{!\ \text{wie } n=2}{=} \underbrace{\mathbb{P}\left[\bigcup_{i=1}^{n} A_i\right]}_{\to \text{Annahme einsetzen}} + \mathbb{P}[A_{n+1}] - \underbrace{\mathbb{P}\left[\bigcup_{i=1}^{n} A_i \cap A_{n+1}\right]}_{①}$$

$$① \ \mathbb{P}\left[\bigcup_{i=1}^{n} (A_i \cap A_{n+1})\right] \overset{!\ \text{Annahme einsetzen}}{=} \sum_{k=1}^{n} (-1)^{k-1} \sum_{i \le i_1 < \dots < i_k \le n} \mathbb{P}[(A_{i_1} \cap A_{n+1}) \cap \dots \cap (A_{i_k} \cap A_{n+1})]$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \sum_{i \le i_1 < \dots < i_k \le n} \mathbb{P}[A_{i_1} \cap \dots \cap A_{i_k} \cap A_{n+1}]$$

$$= \sum_{k=2}^{n+1} (-1)^{k-2} \sum_{i \le i_1 < \dots < i_k \le n+1} \mathbb{P}[A_{i_1} \cap \dots \cap A_{i_{k-1}} \cap A_{i_k}]$$

$$\Rightarrow \mathbb{P}\left[\bigcup_{i=1}^{n+1} A_i\right] = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \dots < i_k \le n} \mathbb{P}[A_{i_1} \cap \dots \cap A_{i_k}] + \mathbb{P}[A_{n+1}]$$

$$+ \sum_{k=2}^{n+1} (-1)^{k-2} \sum_{i \le i_1 < \dots < i_k \le n+1} \mathbb{P}[A_{i_1} \cap \dots \cap A_{i_{k-1}} \cap A_{i_k}]$$

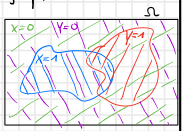
$$= \sum_{k=1}^{n+1} (-1)^{k-1} \sum_{1 \le i_1 < \dots < i_k \le n+1} \mathbb{P}[A_{i_1} \cap \dots \cap A_{i_k}]$$

13/ R.V. auf $\{0,1\}$

Gegeben zwei z.V. X und Y, welche nur die Werte 0 und 1 annehmen können.

Es gilt: $\mathbb{P}[X=0]=\frac{1}{2}$, $\mathbb{P}[Y=0]=\frac{1}{3}$, $\mathbb{P}[X=0, Y=0]=p$.

a) Welche Wert darf p annehmen?



Die Verteilung von $Z=(X,Y)$ ist:

(i) $\mathbb{P}[X=0, Y=0]=p$

(ii) $\mathbb{P}[X=0, Y=1] = \mathbb{P}[X=0] - \mathbb{P}[X=0, Y=0] = \frac{1}{2}-p$

(iii) $\mathbb{P}[X=1, Y=0] = \mathbb{P}[Y=0] - \mathbb{P}[X=0, Y=0] = \frac{1}{3}-p$

(iv) $\mathbb{P}[X=1, Y=1] = \mathbb{P}[X=1] - \mathbb{P}[X=1, Y=0] = \frac{1}{2}-\frac{1}{3}+p = \frac{1}{6}+p$

$\rightarrow$ Da jede $\mathbb{P}:\Omega\rightarrow[0,1]$ muss

$$\left.\begin{array}{l} 0\le p\le 1 \\ 0\le \frac{1}{2}-p\le 1 \rightarrow -\frac{1}{2}\le p\le \frac{1}{2} \\ 0\le \frac{1}{3}-p\le 1 \rightarrow -\frac{2}{3}\le p\le \frac{1}{3} \\ 0\le \frac{1}{6}+p\le 1 \rightarrow -\frac{1}{6}\le p\le \frac{5}{6} \end{array}\right\} \Rightarrow \underline{\underline{0\le p\le \frac{1}{3}}}$$

c) Finden Sie ein Bsp von z.v. U und V so, dass $\mathbb{E}[UV]=\mathbb{E}[U]\mathbb{E}[V]$ aber U und V nicht unabhängig.

$$\begin{cases} U:\Omega\rightarrow\{-1,1\} \text{ mit } \mathbb{P}[U=-1]=\mathbb{P}[U=1]=\frac{1}{2} \\ V=U^3,\ \mathbb{P}[V=-1]=\mathbb{P}[V=1]=\frac{1}{2} \end{cases}$$

$\rightarrow \mathbb{P}[U=1, V=1]=\mathbb{P}[U=1]=\frac{1}{2}\neq\frac{1}{4}=\mathbb{P}[U=1]\cdot\mathbb{P}[V=1] \leftarrow$ nicht unabhängig!

Jedoch: $\mathbb{E}[U]=-1\cdot\frac{1}{2}+1\cdot\frac{1}{2}=0$ $\quad\mathbb{E}[V]=-1\cdot\frac{1}{2}+1\cdot\frac{1}{2}=0$

$$\mathbb{E}[U\cdot V]=-1\,\mathbb{P}[U\cdot V=-1]+1\,\mathbb{P}[UV=1]=-1\cdot\frac{1}{2}+1\cdot\frac{1}{2}=0$$

also Thm 2.37 nicht verletzt da (iii) muss $\forall f,g$ aber wir haben nur $f=X$ und $g=Y$ versucht.

14/ Joint distributions discrete R.V.

Ein Fabrikant verwendet Komponenten A,B,C um Chips herzustellen. Ein Chip besteht aus: A&B oder A&C, beide Möglichkeiten sind gleich Wahrscheinlich. Die Komponenten A,B,C haben X,Y,Z Fehlerstellen, wo $X\sim\text{Poisson}(\lambda)$, $Y\sim\text{Poisson}(\mu)$, $Z\sim\text{Poisson}(2\mu)$. Sei N die Anzahl Fehlerstellen auf dem Chip.

a) Berechne die Verteilung von $X+Y$ und $X+Z$.

$$\mathbb{P}[X+Y=k]=\sum_{j=0}^{k}\mathbb{P}[X=j]\,\mathbb{P}[Y=k-j]=\sum_{j=0}^{k}\frac{\lambda^j}{j!}e^{-\lambda}\cdot\frac{\mu^{k-j}}{(k-j)!}e^{-\mu}=\frac{e^{-(\lambda+\mu)}}{k!}\sum_{j=0}^{k}\frac{k!}{j!(k-j!)}\lambda^j\mu^{k-j}$$

$$=\frac{e^{-(\lambda+\mu)}}{k!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j\mu^{k-j}=\frac{e^{-(\lambda+\mu)}}{k!}(\lambda+\mu)^k \rightarrow (X+Y)\sim\text{Poisson}(\lambda+\mu)$$

$$\mathbb{P}[X+Z=k]=\sum_{j=0}^{k}\mathbb{P}[X=j]\,\mathbb{P}[Z=k-j]=\sum_{j=0}^{k}\frac{\lambda^j e^{-\lambda}}{j!}\frac{(2\mu)^{k-j}}{(k-j)!}e^{-2\mu}\overset{\text{analog}}{=}\dots=\frac{e^{-(\lambda+2\mu)}}{k!}(\lambda+2\mu)^k$$

b) Berechne die Wahrscheinlichkeit, dass ein Chip die Komponente B enthält, falls $N=n\ge 0$

Wir verwenden Satz von Bayes, mit das Ereigniss $D:=\{\text{Chip enthält } B\}$

$$\mathbb{P}[D\mid N=n]=\frac{\mathbb{P}[N=n\mid D]\,\mathbb{P}[D]}{\mathbb{P}[N=n\mid D]\mathbb{P}[D]+\mathbb{P}[N=n\mid D^c]+\mathbb{P}[D^c]}=\frac{\mathbb{P}[X+Y=n]\cdot\cancel{\frac{1}{2}}}{\mathbb{P}[X+Y=n]\cancel{\frac{1}{2}}+\mathbb{P}[X+Z=n]\cancel{\frac{1}{2}}}$$

$$=\frac{n!}{n!}\frac{e^{-(\lambda+\mu)}(\lambda+\mu)^n}{e^{-(\lambda+\mu)}(\lambda+\mu)^n+e^{-(\lambda+2\mu)}(\lambda+2\mu)^n}=\frac{1}{1+e^{-\mu}\left(\frac{\lambda+2\mu}{\lambda+\mu}\right)^n}$$

c) Berechne die Erwartungswert von N

Es gilt $N=(X+Y)1_D+(X+Z)1_{D^c}$, da $X+Y$ und 1_D sowie $X+Z$ und 1_{D^c} unabhängig sind, gilt wegen Thm 2.37 (iii) und linearität von $\mathbb{E}(X)$:

$$\mathbb{E}(N)=\mathbb{E}(X+Y)\mathbb{E}(1_D)+\mathbb{E}(X+Z)\mathbb{E}(1_{D^c})\overset{\mathbb{E}(\text{poisson})}{=}\mathbb{E}(X+Y)\underbrace{\mathbb{P}(1_D)}_{=\frac{1}{2}}+\mathbb{E}(X+Z)\underbrace{\mathbb{P}(1_{D^c})}_{=\frac{1}{2}}$$

$$=\frac{1}{2}(\lambda+\mu+\lambda+2\mu)=\lambda+\frac{3}{2}\mu$$

Or use Formula of total prob and calculate explicitly

$$\mathbb{P}[N=n]=\mathbb{P}[N=n\mid D]\cdot\mathbb{P}[D]+\mathbb{P}[N=n\mid D^c]\,\mathbb{P}[D^c]\rightsquigarrow\mathbb{E}[N]=\sum_{n=0}^{\infty}n\,\mathbb{P}[N=n]$$

15/ Markierte Fische | Maximum-Likelihood-Method

Ziel: Anzahl Fische N in einem Teich zu schätzen.

Vorgehensweise:
i) 10 Fische einfangen, markieren und wieder in den Teich liegen.
ii) Warten bis Fische gut vermischt sind.
iii) Fängen 15 Fische zufällig ein.
iv) Von den 15, X sind Markiert, $15-X$ nicht Markiert

↳ Bestimme die Wahrscheinlichkeitsverteilung P_N von X als Funktion von N.
D.h. was ist die Wahrscheinlichkeit, dass es N Fische gibt, für ein gegebenen X?

Es gibt $\binom{N}{15}$ Möglichkeiten, 15 Fische zu fangen.

Die Anzahl Möglichkeiten, dass davon $k\in\{0,1,\dots,10\}$ Markiert sind, ist $\underbrace{\binom{10}{k}}_{k \text{ aus die Markierte}}\cdot\underbrace{\binom{N-10}{15-k}}_{15-k \text{ aus alle anderen}}$

Nach dem Laplace Modell gilt daher:

$$P_N(X=k)=\frac{\binom{10}{k}\binom{N-10}{15-k}}{\binom{N}{15}}$$

Dann für ein konkretes k, maximiert man $P_N(X=4)$. Um das zu machen, benutzt man den Ausdruck $g(N)=\frac{P_N(X=4)}{P_N(X=4)}$. Für $g(N)=1$ ist P_N gerade bei ein Maxim.

<u>Oder:</u>

Sei M die korrekte Anzahl Fische. Die Proportion von markierte Fische ist dann $\frac{10}{M}$ (10 Fische aus M sind markiert).

Wenn z.B. $X=4$, die Proportion von markierte Fische in unsere Sample von 15 Fische ist $\frac{4}{15}$. Die Proportion von Sample und von Gesamt sollten ungefähr gleich sein d.h. $\frac{10}{M}=\frac{4}{15}$ und M sollte dann $\frac{150}{4}\sim 38$ sein.