Signal and Systems 2

Notation

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

$\mathbb{N} = \{0, 1, 2, \dots\}$

$\mathbb{R}^n$: Euclidean (vector) space of dim n.

$\mathbb{R}^{n\times m}$: matrices with n rows and m colums

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & & a_{2m} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & & a_{nm} \end{bmatrix} \in \mathbb{R}^{n\times m}$$

$s = \underbrace{s_1}_{Re(s)} + j\underbrace{s_2}_{Im[s]} \in \mathbb{C}$

$\left.\begin{array}{l} |s| = \sqrt{s_1^2 + s_2^2} \\ \angle s = \tan^{-1}\left(\frac{s_2}{s_1}\right) \end{array}\right\} s = |s| e^{j\angle s}$

$e^{j\theta} = \cos\theta \, j \sin\theta$

Intervals: $[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$, $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$

$\mathbb{R}_+ = [0, \infty)$

Functions: Discrete: $u(\cdot): \mathbb{N} \to \mathbb{R}^m \quad k \mapsto u(k) = u_k$

Continuous: $x(\cdot): \mathbb{R}_+ \to \mathbb{R}^n \quad t \mapsto x(t)$

Linearity: $\forall x_1, x_2 \in \mathbb{R}^n, a_1, a_2 \in \mathbb{R}$: $f(a_1x_1 + a_2x_2) = a_1 f(x_1) + a_2 f(x_2)$

Laplace transform: $U(s) = \mathcal{L}\{u(\cdot)\}(s) = \int_{-\infty}^{\infty} u(t) e^{-st} \, dt$

one sided: we are only interested in $t \in \mathbb{R}_+ \to U(s) = \int_0^\infty u(t) e^{-st} \, dt$

Convolution: $(u * h)(t) = \int_{-\infty}^{\infty} u(t-\tau) h(\tau) \, d\tau$

one sided: $u * h(t) = \int_0^\infty u(t-\tau) h(\tau) \, d\tau$

Dirac function: $\delta(t) = \begin{cases} \infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$ (not a function but an abstraction)

if $:= \Leftarrow$

iff $:= \Leftrightarrow$

Eigenvalue decomposition:

E-Values: $\det(\lambda I - A) = 0 \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

E-Vectors: $(\lambda_i I - A) w_i = 0$ (linear dependent eqn)

$W = [w_1 \cdots w_n]$ ← doesn't need to be normal

Linear Algebra

- Matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- Trace $\mathrm{tr}(A) = a + d = \lambda_1 + \lambda_2$
- Determinant $\det(A) = ad - bc = \lambda_1 \times \lambda_2$
- Inverse A^{-1} (slide 2.11 in the lecture notes): $A^{-1}A = I$, $AA^{-1} = I$, $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- Eigenvalues λ and eigenvectors v (slide 2.15): $Av = \lambda v$
- Range (slide 2.9): range(A) =span{columns of A}
- Null-space (slide 2.10): null$(A) = \{v \mid Av = 0\}$

- Associative: $(AB)C = A(BC)$
- Distributive: $A(B+C) = AB + AC$
- Non-Commutative: $AB \neq BA$
- Transpose: $(AB)^T = B^T A^T$
- For square $(n\times n)$ matrices: $A \cdot I = I \cdot A = A$
- $\det(AB) = \det(A) \cdot \det(B)$

- **Matrix exponential** $e^{At} = \sum_{i=0}^{\infty} \frac{A^i t^i}{i!}$.
- **Diagonalizable matrix** If $A \in \mathbb{R}^{n\times n}$ can be rewritten as $A = W\Lambda W^{-1}$, where $\Lambda = \mathrm{diag}\{\lambda_i\}_{i=1}^n$. Then, it holds that $e^{At} = We^{\Lambda t}W^{-1}$, and $e^{\Lambda t} = \mathrm{diag}\{e^{\lambda_i}\}_{i=1}^n$, where $\lambda_i \in \mathbb{R}$ are eigenvalues of matrix A and W is matrix with eigenvector columns of A.
- **Nilpotent matrix** $A \in \mathbb{R}^{n\times n}$ is called nilpotent if there exists $k \in \mathbb{N}$ such that $A^k = 0$.
- **Characteristic polynomial of matrix** $p_A = \det(sI - A) = \Pi_{i=1}^n (s - \lambda_i)$, where λ_i are eigenvalues of A.
- **Cayley-Hamilton** Every matrix $A \in \mathbb{R}^{n\times n}$ satisfies its characteristic polynomial

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = 0.$$

- **Solution of a system of linear equations** Given the system of linear equations in matrix form $Ax = y$, $A \in \mathbb{R}^{n\times m}$, $y \in \mathbb{R}^n$ and $x \in \mathbb{R}^m$ is unknown then (slides 2.13):
  - $m = n$: unique solutions if and only if of A is invertible,
  - A is singular: infinitely many or no solutions,
  - $n > m$ (equations > unknowns): generally no solution. If A has rank m, then $x = (A^T A)^{-1} A^T y$ is the unique minimizer of $||Ax - y||$.
  - $n < m$ (equations < unknowns): generally infinite many solutions. If A has rank n, then the system has infinitely many solution. The one with the minimum norm is $x = A^T (AA^T)^{-1} y$.
- **Lipschitz continuous function** A funtion $f : \mathbb{R}^n \to \mathbb{R}^n$ is called Lipschitz if there exists $\lambda > 0$ such that for all $x, \hat{x} \in \mathbb{R}^n$

$$||f(x) - f(\hat{x})|| \leq \lambda ||x - \hat{x}||$$

- **Existence and uniqueness of ODE solution** If in ODE $\dot{x}(t) = f(x(t))$, $f(x(t))$ is Lipschitz continuous function, then there exists a unique solution of the ODE.

$||A(x - \hat{x})|| \leq ||A|| \, ||x - \hat{x}||$ $\qquad ||A|| = \sup_{||v||=1} \frac{||Av||}{||v||}$

$\frac{n!}{(n-j)!\,j!} = \binom{n}{j}$

$\sum_{j=0}^{n} \binom{n}{j} t_1^{n-j} t_2^j = (t_1 + t_2)^n$

Inverse: A is invertible and has a unique inverse iff $\det(A) \neq 0$.
$\Leftrightarrow$ The system of linear eqn $Ax = y$ has a unique solution. $x \in \mathbb{R}^n \; \forall y \in \mathbb{R}^n$
$\Leftrightarrow$ null(A) = Kern$(A) = \{0\}$ $\Leftrightarrow$ range$(A) = \mathbb{R}^n$ $\Leftrightarrow$ Eigenvalues are non-zero.
$\Leftrightarrow \det(A^{-1}) = \frac{1}{\det(A)}$

Otherwise it is singular.

$A^{-1}A = AA^{-1} = I$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{ad - bc}$ or $\left[A \mid I\right] \xrightarrow{\text{Gauss}} \left[I \mid A^{-1}\right]$ or $A^{-1} \overset{\frac{1}{\det(A)}\mathrm{Adj}(A)}{=} \frac{1}{\det(A)} \cdot \begin{bmatrix} ei-fh & ch-bi & bf-ec \\ fg-di & ai-cg & cd-af \\ dh-eg & bg-ah & ae-bd \end{bmatrix}$

Orthogonal: A is orthogonal if $AA^T = A^T A = I$ $\Leftrightarrow$ all columns are orthonormal
$\Leftrightarrow ||Ax|| = ||x||$

Eigenvector: w is an $\overset{EV}{\text{eigenvector}}$ of A if $\exists \lambda$ $\overset{EW}{\text{(eigenvalue)}}$ s.t. $Aw = \lambda w$.
↳ $n\times n$ Matrix has n eigenvalues called the spectrum of A
↳ We find the EW by calculating the roots of the characteristic polynomial:
$\det(A - \lambda I) = 0$ (or $\det(\lambda I - A) = 0$)
↳ We then find the EV w by solving $(\lambda_i I - A) w = 0$

Eigenvalue decomposition:
(Diagonalizable)

$A = W \Lambda W^{-1}$ where $\Lambda = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \lambda_2 & 0 \\ 0 & 0 \cdots & \lambda_n \end{bmatrix}$

$W = [w_1 \; w_2 \cdots w_n]$

A is diagonalizable if W is invertible.
If EW λ_i are distinct ($\lambda_i \neq \lambda_j, \forall i \neq j$) then $\underbrace{\text{its EV are linearly independant}}_{\text{W is invertible}}$
If EW λ_i aren't distinct, then we dont know.

Cayley-Hamilton: Every matrix $A \in \mathbb{R}^{n\times n}$ satisfies its characteristic polynomial:
$A^n + \lambda_1 A^{n-1} + \lambda_2 A^{n-2} + \dots + \lambda_n I = 0$

Nilpotent Matrices: A is nilpotent if $\exists k \in \mathbb{N}$ s.t. $A^k = 0. \Rightarrow A^n = 0 \quad \forall n \geq k$
$\Leftrightarrow A$ nilpotent iff all EW are equal to zero.

Symmetric: $A = A^T$
Symmetric Matrices have $\mathbb{R}$-EW and orthogonal eigenvectors.

Positive definit: $x^T A x > 0 \; \forall x \neq 0$ or $\lambda_i > 0$
$(A > 0)$

Positiv semi definit: $x^T A x \geq 0$ or $\lambda_i \geq 0$
$(A \geq 0)$

Singular value decomposition:
For any $A \in \mathbb{R}^{n\times m}$, $A = U \Sigma V^T$ where $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ are orthogonal and $\Sigma \in \mathbb{R}^{n\times m}$ diagonal with non-negative elements. $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$

ODE:
given $\dot{x}(t) = Ax(t)$ we know there exists a unique solution since $f(x(t))$ is Lipschitz.
it follows: $\forall t_1, t_2 \in \mathbb{R}_+ \quad e^{A(t_1+t_2)} = e^{At_1} e^{At_2}$

For any two commutable A_1 and A_2 (i.e. $A_1 A_2 = A_2 A_1$) then $e^{(A_1+A_2)t} = e^{A_1 t} e^{A_2 t}$

More properties in serie 2

Modeling Systems

Steps toward modeling

Step I: • Identify input variables $u(t) \in \mathbb{R}^m$ → things you can control

• Identify output variable $y(t) \in \mathbb{R}^p$ → things you can measure

• Define the state $x(t) \in \mathbb{R}^n$ that you wish to model

Mechanical:

* states: position x, velocity $\dot{x}$.
* equations: Newton law $F = m\ddot{x}$, friction and damper $F = -\rho\dot{x}$, spring $F = -kx$.
* energy: kinetic $E_{kin} = \frac{1}{2}m\dot{x}^2$, potential $E_{pot} = \frac{1}{2}mx^2$, spring $E_{dyn} = \frac{1}{2}kx^2$.

Circuits:

* states: currents i, voltages v.
* equations: Kirchhof laws, resistance $v_R(t) = Ri_R(t)$, capacitance $C\frac{dv_C(t)}{dt} = i_C(t)$, inductance $L\frac{di_L(t)}{dt} = v_L(t)$.
* energy: capacitance $E_C = \frac{1}{2}Cv_C^2$, inductance $E_L = \frac{1}{2}Li_L^2$.

on the exam, we are usually given the state

Step II: • Take derivative of state and use first principle to write equation

↳ Newton laws: $\sum \text{Forces} = m\ddot{x}$

$\sum \text{torques} = J\ddot{\theta}$

Kirchof laws: $C\frac{dv_c(t)}{dt} = i_c(t)$

$L\frac{d\,i_c(t)}{dt} = V_L(t)$

$\sum$ current in/out of a node $=0$

$\sum$ voltages around closed loop $=0$

Step III: write everything in state-space form

$\dot{x}(t) = f(x(t), u(t), t)$

$y(t) = f(x(t), u(t))$

⚠ not necessarily $\dot{x}(t) = Ax + Bu$, $y = Cx + Du$

E_{tot} is of the form $\frac{1}{2}x^T Q x$

$P = \frac{dE(t)}{dt} = \frac{1}{2}\dot{x}^T Q x + \frac{1}{2}x^T Q \dot{x}$

Examples torque and Forces at the end!

State space representation

Elements: time: $t \geq 0$

state: $x(t) \in \mathbb{R}^n$

output: $y(t) \in \mathbb{R}^p$

input: $u(t) \in \mathbb{R}^m$

Order/Dimension of the system $= n$ (Number of states)

Dynamics: $\frac{dx(t)}{dt}\ \dot{x}(t) = f(x(t), u(t), t) = f(x,u,t) = \begin{bmatrix} f_1(x,u,t) \\ \vdots \\ f_n(x,u,t) \end{bmatrix}$

vector field

$y(t) = h(x(t), u(t), t) = h(x,u,t) = \begin{bmatrix} h_1(x,u,t) \\ \vdots \\ h_p(x,u,t) \end{bmatrix}$

Linear: The functions f and h can be written in the form:

$\dot{x}(t) = A(t)x(t) + B(t)u(t)$, $y(t) = C(t)x(t) + D(t)u(t)$

Time invariant: The dynamics do not depend explicitly on time:

$\dot{x}(t) = f(x(t), u(t))$ $y(t) = h(x(t), u(t))$

Autonomous: A system is autonomous if time invariant + has no input variables

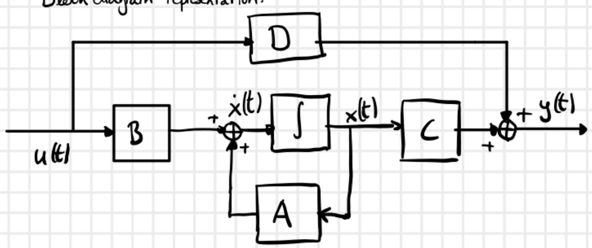
$\dot{x}(t) = f(x(t))$ $y(t) = h(x(t))$

For LTI systems: $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$

$C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$

$$\begin{cases} \dot{x}(t) = Ax + Bu \\ y(t) = Cx + Du \end{cases}$$

Block diagram representation:



The state space representation is composed of:

- n coupled, first order, linear differential equations
- p linear algebric equations
- Time invariant coefficients

$$\begin{aligned}
\dot{x}_1(t) &= a_{11}x_1(t) + \dots + a_{1n}x_n(t) + b_{11}u_1(t) + \dots + b_{1m}u_m(t) \\
\dot{x}_2(t) &= a_{21}x_1(t) + \dots + a_{2n}x_n(t) + b_{21}u_1(t) + \dots + b_{2m}u_m(t) \\
&\vdots \\
\dot{x}_n(t) &= a_{n1}x_1(t) + \dots + a_{nn}x_n(t) + b_{n1}u_1(t) + \dots + b_{nm}u_m(t) \\
y_1(t) &= c_{11}x_1(t) + \dots + c_{1n}x_n(t) + d_{11}u_1(t) + \dots + d_{1m}u_m(t) \\
y_2(t) &= c_{21}x_1(t) + \dots + c_{2n}x_n(t) + d_{21}u_1(t) + \dots + d_{2m}u_m(t) \\
&\vdots \\
y_p(t) &= c_{p1}x_1(t) + \dots + c_{pn}x_n(t) + d_{p1}u_1(t) + \dots + d_{pm}u_m(t)
\end{aligned}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{p1} & \cdots & c_{pn} \end{bmatrix} \quad D = \begin{bmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & \ddots & \vdots \\ d_{p1} & \cdots & d_{pm} \end{bmatrix}$$

Higher order ODE: We can convert any order of diff. eqn by defining lower order derivatives as states.

Time invariant: The explicit time dependance can be eliminated by introducing an additional state $x_i = t$, $\dot{x}_i = 1$

Linearization: → used for non-linear systems

$$\dot{x} = f(x_0, u_0) + \frac{\partial f}{\partial x}\bigg|_{(x_0,u_0)}(x - x_0) + \frac{\partial f}{\partial u}\bigg|_{(x_0,u_0)}(u - u_0)$$

where (x_0, u_0) usually is the steady state (see s1a24)

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1}(\hat{x},\hat{u}) & \cdots & \frac{\partial f_1}{\partial u_m}(\hat{x},\hat{u}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1}(\hat{x},\hat{u}) & \cdots & \frac{\partial f_n}{\partial u_m}(\hat{x},\hat{u}) \end{bmatrix}$$

Coordinate transformation:

Restriction on LTI systems and linear change of coordinates: $\hat{x} = Tx$, $T \in \mathbb{R}^{n\times n}$, $\det(T) \neq 0$. Then both systems are equivalent.

In the new coordinates we get:

$$\dot{\hat{x}}(t) = TAT^{-1}\hat{x}(t) + TBu(t)$$
$$y(t) = CT^{-1}\hat{x}(t) + Du(t)$$

A and $\hat{A}$ are similar, i.e. they have the same rank, e-values $\text{spec}[A] = \text{spec}[\hat{A}]$ and charac. poly.. $\Rightarrow$ They have the same properties for controllability and observability.

↳ $\tilde{A} = TAT^{-1} = TW\Lambda W^{-1}T^{-1} = \tilde{W}\Lambda\tilde{W}^{-1}$, $\tilde{W} = TW$

example: see S1a3.2, S3a1.1, S5a3.1, nonlinear: S9a4

Solution of state space equations

A pair of functions $x(\cdot):[t_0,t_1] \to \mathbb{R}^n$, $y(\cdot):[t_0,t_1] \to \mathbb{R}^p$ is a solution of the state space system over the interval $[t_0,t_1]$ starting at $x_0 \in \mathbb{R}^n$ if

1. $x(t_0) = x_0$
2. $\dot{x}(t)\ f(x(t)) \quad \forall t \in [t_0,t_1]$
3. $y(t) = h(x(t)) \quad \forall t \in [t_0,t_1]$

Existance and uniqueness of solutions

For a given Problem, we ask ourselves the questions of:

existance → no / for some interval / for all t

Uniqueness → yes (one solution) / no (many solutions)

> Thm: If f is Lipschitz, then the differential equation $\dot{x} = f(x(t))$, with initial condition $x_0 \in \mathbb{R}^n$ has a unique solution $x(\cdot):[0,T] \to \mathbb{R}^n$ for all $T \geq 0$ and all $x_0 \in \mathbb{R}^n$.

In other words: state space systems defined by Lipschitz vector fields are well behaved.
↳ They have unique solutions, over all time, whenever they start.

> Thm: If f is Lipschitz then the solutions starting at $x_0, \hat{x}_0 \in \mathbb{R}^n$ are such that for all $t \geq 0$: $\|x(t) - \hat{x}(t)\| \leq e^{\lambda t}\|x_0 - \hat{x}_0\|$.

But, even if a solution exists, doesn't mean we can find it.

For non autonomous systems, in order to have a solution we need:

- $f(x,u,t)$ Lipschitz in x, continuous in u and t
- $u(t)$ continuous for "almost all" t

Following theses points, we can proove that:

> LTI systems have unique solutions

Lipschitz:

- a function $f:\mathbb{R}^n \to \mathbb{R}^n$ is called Lipschitz if $\exists\ \lambda > 0$ s.t. $\forall\ x, y \in \mathbb{R}^n$: $\|f(x) - f(y)\| \leq \lambda\|x - y\|$
- If $\frac{df}{dx}(t) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \cdots & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$ is bounded then f is Lipschitz
- Lipschitz $\leq$ continuous, Lipschitz ~ doesn't grow faster than linear.

Solution LTI systems

> state: $x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$
>
> output: $y(t) = C\Phi(t)x_0 + \int_0^t C\Phi(t-\tau)Bu(\tau)d\tau + Du(t)$
>
> state transition Matrix: $\Phi(t) = e^{At} = I + At + \frac{A^2t^2}{2!} + \dots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$

Rmk: - the integral is computed element by element.

- The state transition matrix ($\Phi(t)$) is such that
  1. $\Phi(0) = I$
  2. $\frac{d}{dt}\Phi(t) = A\Phi(t)$
  3. $\Phi(-t) = [\Phi(t)]^{-1}$
  4. $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2)$
- We can decompose the state solution in 2 parts:

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

Total transition = Zero Input transition + Zero State transition

$u(t) = 0\ \forall t \Rightarrow x(t) = ZIT$ · Linear Function of initial state x_0

$x_0 = 0 \Rightarrow x(t) = ZST$ · Linear function of input, i.e.:
$u(\tau) = a_1 u_1(\tau) + a_2 u_2(\tau)$ for $\tau \in [0,t]$
$x(t) = a_1 x_1(t) + a_2 x_2(t)$
· Convolution integral

- We can do the same for the output:

$$y(t) = C\Phi(t)x_0 + C\int_0^t \Phi(t-\tau)Bu(\tau)d\tau + Du(t)$$

Total Response = Zero Input Response + Zero State Response

Transition Matrix $\Phi(t)$ computation

We have multiple ways of computing $\Phi(t)$ here are a few of them:

- Diagonalizable Matrices:

  We perform a change of coordinates using EW decomposition.

  $$\Phi(t) = e^{At} = We^{\Lambda t}W^{-1} \quad \text{where} \quad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{bmatrix}$$

- Non diagonalizable Matrices

  - If the Matrix is Nilpotent then we can directly compute the STM:

    $$\Phi(t) = I + At + \dots + \frac{A^{(k-1)}t^{(k-1)}}{(k-1)!} \rightarrow \text{finite series: computable}$$

  - "Nilpotent decomposition":

    If A can be written as $A = D + N$ where $D = \mathrm{diag}\{d_i\}_{i=0}^{n}$, N nilpotent with $\underline{DN = ND}$, then $e^{At} = e^{(D+N)t} = e^{Dt} \cdot e^{Nt} = e^{Dt} \sum_{i=0}^{k-1} \frac{N^i t^i}{i!}$, $N^k = 0$

E-Values and E-Vectors

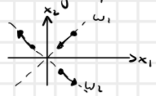
$A w_i = \lambda_i w_i$, $\dot{x}(t) = Ax(t) \rightarrow x(t) = \Phi(t)\, x(0)$

Now if $x(0) = w_i \Rightarrow \dot{x}(0) = Aw_i = \lambda_i w_i$

↳ So if we start on e-vector, we stay on e-vector.

$\|x(t)\|$ increases/decreases depending on the sign of λ

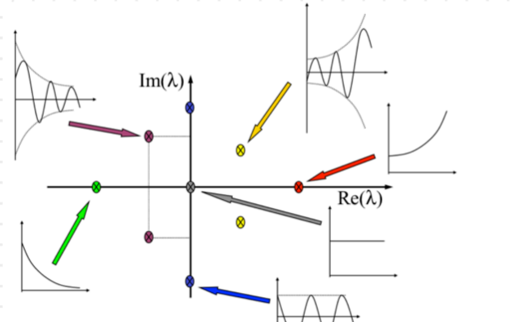
↳ E.g. $n = 2$, $\lambda_1 < 0$, $\lambda_2 > 0$



Phase-plane-plot: see slides 3.27

Rmk: ZIT is a linear combination of terms of the form:
- 1 if $\lambda_i = 0$ $(\sigma = 0 = \omega)$
- $e^{\sigma t}$ if $\lambda_i = \sigma$ $(\sigma \neq 0, \omega = 0)$
- $\sin(\omega t)$ if $\lambda_i = \pm j\omega$ $(\sigma = 0, \omega \neq 0)$
- $e^{\sigma t}\sin(\omega t)$ and $e^{\sigma t}\cos(\omega t)$ if $\lambda_i = \sigma \pm j\omega$ $(\sigma \neq 0, \omega \neq 0)$

• Part of ZIT corresponding to $\lambda_i = \sigma \pm j\omega$
- c if $\sigma = \omega = 0$
- $\rightarrow 0$ if $\sigma < 0$
- Periodic if $\sigma = 0$, $\omega \neq 0$
- $\rightarrow \infty$ if $\sigma > 0$



(Asymptotical) Stability

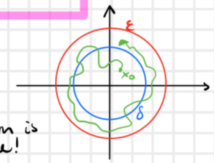
Consider the system $\dot{x}(t) = Ax(t) + Bu(t)$, let $x(t)$ be its ZIT $x(t) = \Phi(t)x_0$

- Let A be diagonalizable:

> The system is called stable if $\forall \varepsilon > 0$ $\exists$ $\delta > 0$ s.t. if $\|x_0\| \leq \delta$ then $\|x(t)\| \leq \varepsilon$ $\forall$ $t \geq 0$
> Otherwise the system is called unstable.

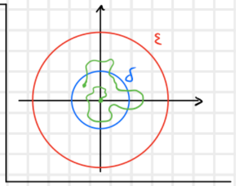
stability game:

1. somebody gives you any ε
2. if I can always choose a δ s.t. when I start inside δ then I'll stay inside $\varepsilon, \forall t.$ $\Big\}\Rightarrow$ system is stable!



> The system is called asymptotically stable if it is stable and in addition $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$

Rmk: For linear systems, stability is defined as a system property. (only one equilibrium point!)
For non-linear systems, stability is defined as an equilibrium property. (multiple equilibrium)



Checking stability (linear systems)

> Thm: For a system with diagonalizable A matrix, it is:
> - Stable iff $\mathrm{Re}[\lambda_i] \leq 0$ $\forall i$
> - Asymptotically stable iff $\mathrm{Re}\,\lambda[i] < 0$ $\forall i$
> - Unstable iff $\exists i: \mathrm{Re}[\lambda_i] > 0$

Now if A isn't diagonalizable (often because EW repeat themselves)
The system is:
- Asymptotically stable if and only if $\mathrm{Re}[\lambda_i] < 0$ $\forall i$
- Unstable if $\exists i: \mathrm{Re}[\lambda_i] > 0$

If $\sigma < 0$:
- $t^k e^{\sigma t}$, $t^k e^{\sigma t} \sin \omega t \xrightarrow{t \to \infty} 0$
- ZIT tends to zero (for some initial states)

If $\sigma > 0$:
- $t^k e^{\sigma t}$, $t^k e^{\sigma t} \sin \omega t \xrightarrow{t \to \infty} \infty$
- Hence ZIT tends to infinity (for some initial state)

If $\sigma = 0$:
- 1, $\cos(\omega t)$, $\sin(\omega t)$ remain bounded
- t^k, $t^k\cos(\omega t)$, $t^k \sin(\omega t) \longrightarrow \infty$ for $k \geq 1$

$\Big\}$ ZIT may remain bounded or tends to infinity, depending on x_0.

- We cannot tell the exact behavior just by looking @ the EW

> For 2nd order polynomials, $a\lambda^2 + b\lambda + c = 0$
> - a, b, c same sign $\Leftrightarrow$ $\mathrm{Re}(\lambda_i) < 0\, \forall i$ • (if a, b or $c = 0 \Rightarrow$ Compute EW)
> - a, b, c not same sign $\Leftrightarrow$ $\exists$ i s.t. $\mathrm{Re}(\lambda_i) > 0$

Zero state transition

We consider the transition with an Impulse $\delta(t)$ for the scalar $(n=m=1)$ case:

$$h(t)=ZST=\int_0^t \Phi(t-\tau)B\,\delta(\tau)\,d\tau = e^{at}\int_0^t e^{-a\tau} b\,\delta(\tau)\,d\tau = e^{at}b$$

In the general scalar case:

$$x(t)=ZST=\int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau=\int_0^t e^{a(t-\tau)} b\,u(\tau)\,d\tau=\int_0^t h(t-\tau)u(\tau)\,d\tau$$

$$=(h*u)(t) \qquad \text{where } h=be^{at}$$

Now we look the general Impulse transition $H(t)$ $n,m\in\mathbb{N}$

$$H(t)=\begin{bmatrix} h_{11}(t) & \cdots & h_{1m}(t) \\ \vdots & & \vdots \\ h_{n1}(t) & \cdots & h_{nm}(t)\end{bmatrix}=\Phi(t)B$$

$\boxed{x(t)=ZST=(H*u)(t)}$ Integral computed element by element.

Output impulse response $K(t)$

output ZSR is: $y(t)=C\int_0^t \Phi(t-\tau)Bu(\tau)\,d\tau + D\,u(t)$

let the input $u(t)=\delta(t)$ then $\boxed{K(t)=C\,\Phi(t)B+D\delta(t)}$

Then the output ZSR is: $\boxed{y(t)=(K*u)(t)}$

$K(t)$ determines the input-output behavior, un affected by the state (S3a1.3)

ZST

Thm: Assume that $Re[\lambda_i]<0$ $\forall i$. Then $\exists$ $\alpha\geq 0$ s.t. ZST, $x(t)$ satisfies: $\|u(t)\|\leq M$ $\forall t\geq 0 \Rightarrow \|x(t)\|\leq \alpha M$ $\forall t\geq 0$

If in addition $u(t)\xrightarrow{t\to\infty}0$ then $x(t)\xrightarrow{t\to\infty}0$

↳ If/when the input goes to zero, so does the state.
small input leads to small states

Stability with inputs

Complete Response = ZIR + ZSR

If $Re(\lambda_i)<0$ $\forall i$

- BIBS: Bounded input bounded state
- BIBO: Bounded input bounded output
- ZIT/ZIR and ZST/ZSR tend to 0 if input tends to 0

Energy, Controllability, Observability

System energy can be represented in the form $E(t)=\frac{1}{2}x^TQx$
where $Q=Q^T>0$.

$$E(t+h)=E(t)+\int_t^{t+h}\frac{dE(t)}{dt}dt=E(t)+\int_t^{t_1}P(t)\,dt$$

System power can be found by derivating the Energy w.r.t. t

$$P(t)=\frac{dE(t)}{dt}=\frac{1}{2}(\dot{x}^TQx+x^TQ\dot{x})=\frac{1}{2}x(t)^T(A^TQ+QA)x(t)+\frac{1}{2}(u(t)^TB^TQx(t)+x(t)^TQBu(t))$$

$$=-\frac{x(t)^TRx(t)}{2} \quad \text{for autonomous systems, i.e. } u=0,\ R=-(A^TQ+QA)$$

For stable system the energy is non-increasing which means $P=-\frac{1}{2}x^TRx<0$
or, in other words, $x^TRx>0$ i.e. $R>0$.

Lyapunov

$\hat{=}$ asymptotically stable

Thm: The e-values of A have $Re(\lambda_i)<0$ $\forall i$ iff $\forall R=R^T>0$ there exists a unique $Q=Q^T>0$ such that $A^TQ+QA=-R$

The Lyapunov equation $A^TQ+QA=-R$ can be rewritten as $\hat{A}q=r$
And if $\hat{A}$ is non-sigular, there is a unique q, which means stability.

Example: Wien Oscillator $R=1, C=1$

$$A=\begin{bmatrix}-1 & 1-k \\ 1 & k-2\end{bmatrix}$$, we choose (rather we are given) $R=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$
and want to find a corresponding Energy
$\hat{=}$ a corresponding Q s.t. $A^TQ+QA=-R$

We set $Q=\begin{bmatrix}q_1 & q_2\\ q_2 & q_3\end{bmatrix}$

$$↳ \begin{bmatrix}-1 & 1\\ 1-k & k-2\end{bmatrix}\begin{bmatrix}q_1 & q_2\\ q_2 & q_3\end{bmatrix}+\begin{bmatrix}q_1 & q_2\\ q_2 & q_3\end{bmatrix}\begin{bmatrix}-1 & 1-k\\ 1 & k-2\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}$$

$$\begin{bmatrix} q_2-q_1+q_2-q_1 & q_3-q_2+q_1(1-k)+q_2(k-2) \\ q_1(1-k)+(k-2)q_2+q_3-q_2 & q_2(1-k)+q_3(k-2)+q_2(1-k)+q_3(k-2)\end{bmatrix}=\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}$$

$$\left.\begin{aligned} q_2-q_1&=-1\\ q_3+q_2(k-3)+q_1(1-k)&=0\\ q_2(1-k)+q_3(k-2)&=-1\end{aligned}\right\}\ \underbrace{\begin{pmatrix}1 & -1 & 0\\ 1-k & k-3 & 1\\ 0 & k-1 & 3-k\end{pmatrix}}_{\hat{A}}\underbrace{\begin{pmatrix}q_1\\ q_2\\ q_3\end{pmatrix}}_{q}=\underbrace{\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}}_{r}$$

Then for different k's

$\underline{k=2}$: $\hat{A}$ non singular $\Rightarrow Q=\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}>0 \Rightarrow$ asymptotically stable

$\underline{k=3}$: $\hat{A}$ sigular $\Rightarrow$ no solution for $Q \Rightarrow$ has at least one EW with non neg. $\mathbb{R}$-part

$\underline{k=4}$: $\hat{A}$ non singular $\Rightarrow Q=\begin{bmatrix}-6 & -7\\ -7 & -11\end{bmatrix}<0$ Negativ definit $\Rightarrow$ unstable
more and more energy.

$\underline{s4a1}$: - given a R, if $\exists!$ $P=P^T>0$, that solves $A^TP+PA=-R$, then $Re\{eig(A)\}<0$

- if the P that we find isn't positiv definit or if there is no P, we can conclude A has @ least one EW with non-negative $\mathbb{R}$-part.

$\underline{s4a4.2}$: - If we can find an energy-function $E(t)$ that satisfies

- $E(t+h)\leq E(t)$ $\quad h\geq 0$
- $\lim_{t\to\infty} E(x(t))=0$

then the system is stable.

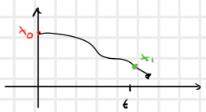
Rmk: - Connection to non-linear systems:
Given some $R>0$, the matrix Q solving the above defines a Lyapunov function $V(x)=\frac{1}{2}x^TQx$ which can be used to prove stability using Lyapunov methods from nonlinear notes
- It is actually enough to check that one R has a unique Q,
(it is then somehow generalized ...)

Controllability

$\boxed{\text{A system is called controllable over } [0,t] \text{ if } \forall x(0)=x_0, \forall x \text{ there is an input s.t. } x(t)=x_1.}$

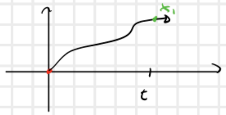
Following statements are equivalent:

controllable over $[0,t]$



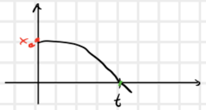
$\Longleftrightarrow$

$\forall x_1, \exists$ input s.t.
$x(0)=0$ and $x(t)=x_1$



$\Longleftrightarrow$

$\forall x_0, \exists$ input s.t.
$x(t)=0$



$\Longleftrightarrow$

$$\boxed{P=[B \;\; AB \;\; \cdots \;\; A^{n-1}B] \text{ has full rank } (=n)}$$

$\Longleftrightarrow$

$$W_c(t)=\int_0^t e^{A\tau}BB^Te^{A^T\tau}\,d\tau \quad \text{is invertible}$$

Rmk: - If not controllable, the "reachable states" are given by the range (P)

- The time it takes to do that is arbitrary!
- But the faster we go, the more energy and the bigger the input we need.
- Rank $[\lambda_i I - A \quad B] < n$ for some λ_i, then the mode corresponding to λ_i is uncontrollable. If this λ_i has $Re[\lambda_i]<0$, then the mode is stabilizable.
- If controllable, it is always possible to find a matrix K s.t. $A+BK$ (closed loop controller) is asymptotically stable

Minimum energy inputs: Assume that the system is controllable. Given $x_1 \in \mathbb{R}^n$ and $t>0$, the input that drives the system from $x(0)=0$ to $x(t)=x_1$ and has the minimum energy is given by:

$$u_m(\tau)=B^Te^{A^T(t-\tau)}W_c(t)^{-1}x_1 \;, \text{ for } \tau\in[0,t]$$

e.g. $u_m(t)=B^Te^{A^T(1-t)}W_c(1)^{-1}x_1$, for $t\in[0,1]$

s5a2(4): Furthermore, x_1 has to be in the reachable set of the system $\hat{=}$ range(P)

Observability:

$\boxed{\text{System is observable over } [0,t] \text{ if given } u:[0,t]\to\mathbb{R}^m, \; y:[0,t]\to\mathbb{R}^p \text{ we can uniquely determine } x:[0,t]\to\mathbb{R}}$

$$\Longleftrightarrow \quad \boxed{Q=\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1}\end{bmatrix} \text{ has rank } n}$$

$$\Longleftrightarrow \quad W_o(t)=\int_0^t e^{A^T\tau}C^TCe^{A\tau}\,d\tau \quad \text{is invertible}$$

Rmk: - if not observable, the set of unobservable states is given by $Null(Q)=\{x \mid Qx=0\}$

- $x\in\mathbb{R}^n$ unobservable iff $Ce^{A\tau}x=0 \;\; \forall \tau\in[0,t]$
- $x=0$ is always unobservable. Only if it is the only one, then observable
- observability is time invariant.
- Rank $\begin{bmatrix}\lambda_i I - A \\ C\end{bmatrix} < n$ for some λ_i, then the corresponding mode is unobservable. If this unobservable mode has $Re[\lambda_i]<0$ then it is detectable.

Error dynamics $e=x-\tilde{x}(t) \Rightarrow \dot{e}(t)=(A-LC)e(t)$ (4,36)

Output derivative interpretation: $\rightsquigarrow$ see slides ... Derivative of signal is useless

Observers If system is observable, (4.33 | 4.37) we can build an observer (L s.t. system error $\xrightarrow{t\to\infty} 0$

Detectability: $\boxed{\text{The system is detectable if all eigenvalues of } \hat{A}_{22} \text{ and } \hat{A}_{44} \text{ in the Kalman decomposition have negative real part.}}$

Or: detectable $\Longleftrightarrow$ unobservable modes are asymptotically stable
If the system is detectable, we can build an observer for the observable part with error decaying to zero (hence the condition that modes are asymptotically stable).

Stabilizable: $\boxed{\text{The system is stabilizable if all eigenvalues of } \hat{A}_{33} \text{ and } \hat{A}_{44} \text{ in Kalmann decomposition have negative real part.}}$

Or: uncontrollable states are asymptotically stable.
Such that we can select the input (design a controller) such that the overall system will be asymptotically stable.

Observable $\Rightarrow$ Detectable : if unobserv. states go to zero

Controllable $\Rightarrow$ Stabilizable : if uncontrollable states go to zero.

Hautus - Test: • System stabilizable $\Longleftrightarrow$ $\forall$ E-values λ_i with $Re\{\lambda_i\}>0$, the matrix $(\lambda_i I - A \quad B)$ must have rank n.

• System detectable $\Longleftrightarrow$ $\forall$ E-values λ_i with $Re\{\lambda_i\}>0$ the matrix $\begin{pmatrix}\lambda_i I - A \\ C\end{pmatrix}$ must have rank n.

Kalman Decomposition:

Nothing but a fancy change of coordinates

$$\hat{x}(t)=Tx(t)=\begin{bmatrix}\hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \\ \hat{x}_4(t)\end{bmatrix} \begin{matrix} \leftarrow \text{controllable \& observable} \\ \leftarrow \text{controllable \& unobservable} \\ \leftarrow \text{uncontrol. \& observ.} \\ \leftarrow \text{uncontrol. \& unobserv.}\end{matrix}$$

For a given system, there may not exist all combinations

$$\hat{A}=TAT^{-1}=\begin{bmatrix}\hat{A}_{11} & 0 & \hat{A}_{13} & 0 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} \\ 0 & 0 & \hat{A}_{33} & 0 \\ 0 & 0 & \hat{A}_{43} & \hat{A}_{44}\end{bmatrix}$$

$$\hat{B}=TB=\begin{bmatrix}\hat{B}_1 \\ \hat{B}_2 \\ 0 \\ 0\end{bmatrix} \qquad \hat{C}=CT^{-1}=[\hat{C}_1 \;\; 0 \;\; \hat{C}_3 \;\; 0]$$

We can use this coordinate transform to check stabilizability and detectability. For those properties, we are interested in the e-values of the sub-matrices.
As known from coordinate Transform,

$$spec[A]=spec[\hat{A}]=\underbrace{spec[\hat{A}_{11}] \cup spec[\hat{A}_{22}]}_{\text{controllable}} \cup \underbrace{spec[\hat{A}_{33}] \cup spec[\hat{A}_{44}]}_{\text{uncontrollable}}$$

observable

unobservable

Example:
$$\dot{x}(t) = \begin{bmatrix} -2 & 1 \\ -3 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Determine if the system is controllable, observable, if not what modes are detectable \ stabilizable.

$\text{spec}[A] = \{-1, 1\} \rightarrow$ not stable!

$Q = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \leadsto$ Full Rank $\Rightarrow$ system is observable $\Rightarrow \text{Spec}[\hat{A}_{22}] = \text{spec}[\hat{A}_{44}] = \emptyset$

$\Rightarrow$ system is also detectable

$P = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \leadsto \text{rank}(P) = 1 < 2 = n \Rightarrow$ system is uncontrollable $\Rightarrow \text{spec}[\hat{A}_{33}] \neq \emptyset$

but what is $\text{spec}[\hat{A}_{33}] \overset{?}{=} \{-1\} | \{1\} | \cancel{\{1, -1\}}$

We can eliminate 3rd option since $\text{rank}(P) = 1$ so the reachable set is of Dim 1.

Now, to determine which mode is uncontrollable we can use the test:

$$\text{Rank}[\lambda I - A \quad B] \begin{cases} \lambda = 1 & \begin{bmatrix} \lambda+2 & -1 & 1 \\ 3 & \lambda-2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \leadsto \text{Rank} = 1 < 2 \Rightarrow \text{spec}[\hat{A}_{33}] = \{1\} \\ \lambda = -1 & \begin{bmatrix} \lambda+2 & -1 & 1 \\ 3 & \lambda-2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 1 \end{bmatrix} \leadsto \text{Rank} = 2 \Rightarrow \text{spec}[\hat{A}_{11}] = \{-1\} \end{cases}$$

As a complement: we can find that $T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, then $\dot{\hat{x}} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$

$\hat{A}_{11}$, $\hat{A}_{13}$, $\hat{A}_{31}$, $\hat{A}_{33}$, B_1, B_3

$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{x}(t)$,

$\hat{C}_1$, $\hat{C}_3 = 0$

only the observable sub-space

$\dot{\hat{x}}_2 = \hat{x}_2 \rightarrow \hat{x}_2 = e^{t} x_2(0) \rightarrow \pm\infty$ if $x_2(0) \neq 0$ $\longrightarrow$ unstable, and can't affect it with input: i.e. not stabilizable!

$\dot{\hat{x}}_1 = -\hat{x}_1(t) + \hat{x}_2(t) + u(t)$ can steer with $u(t)$

e.g. $u(t) = -\hat{x}_2(t)$ Feedback Controller $\Rightarrow \dot{\hat{x}}_1(t) = -\hat{x}_1(t) \Rightarrow \hat{x}_1(t) = e^{-t} x_1(0)$ stable

$\leadsto$ If one was to modify the system by removing some states, we can usually see it isn't a good idea if some of the properties (stability, controllability, observability) are changed.

Continuous LTI systems, frequency domain

Laplace transform

$$f(t): \mathbb{R} \rightarrow \mathbb{R} \quad \underset{\mathcal{L}^{-1}\{\cdot\}}{\overset{\mathcal{L}\{\cdot\}}{\rightleftarrows}} \quad F(s): \mathbb{C} \rightarrow \mathbb{C}$$
$$t \rightarrow f(t)$$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt, \quad \mathcal{L}^{-1}\{F \qquad s = \sigma + j\omega$$

Rmk: - we assume that: - $f(t) = 0 \ \forall t < 0$

- $f(t)$ is absolute integrable: $f(t) e^{-st} \xrightarrow{t \to \infty} 0$

- Can be defined for matrix-valued functions, by taking the integral elem by elem

$f: \mathbb{R} \rightarrow \mathbb{R}^{n \times m} \qquad F: \mathbb{C} \rightarrow \mathbb{C}^{n \times m}$

Properties:

i) Linear: $\mathcal{L}\{a f_1 + b f_2\} = a F_1 + b F_2$

ii) shift: $\mathcal{L}\{f(t) e^{-at}\} = F(s+a)$, $\mathcal{L}\{f(t-a)\}(s) = e^{-as} \mathcal{L}\{f(t)\}$ $a > 0$

iii) Time derivative: $\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = F(s) \cdot s - f(0)$

iv) Convolution: $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau$

$$\mathcal{L}\{(f*g)(t)\} = F(s)\, G(s)$$

Usefull transformations:

$\mathcal{L}\{\delta(t)\} = 1$

$\mathcal{L}\{1\} = \frac{1}{s}$

$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$

$\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2} \qquad \mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}$

$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \qquad \mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - s f(0) - f'(0)$

$\mathcal{L}\left\{\frac{e^{at} - e^{bt}}{a-b}\right\} = \frac{1}{(s-a)(s-b)}$

more at the end.

Use Partial fraction decomposition to compute inverse Laplace transform

Initial value theorem: $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$

Final value theorem: $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

(can only be used when poles of $F(s)$ are either in the LHP or $F(s)$ has at most a single pole at the origin, i.e. when both limit exist)

~ Use when $f(t)$ is asymptotically stable or stable with a simple $\lambda_i = 0$

$\cancel{\lambda_i = \pm j\omega}$, $\cancel{Re(\lambda_i) > 0}$

LTI system in Laplace domain

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \mathcal{L}\{Ax(t) + Bu(t)\} \Rightarrow sX(s) - x(0) = AX(s) + BU(s) \Rightarrow$$

$$\boxed{\begin{aligned} X(s) &= (sI - A)^{-1} x_0 + (sI - A)^{-1} B\, U(s) \\ Y(s) &= C X(s) + D U(s) \end{aligned}}$$

$X(s) \in \mathbb{C}^n, U(s) \in \mathbb{C}^m, Y(s) \in \mathbb{C}^p, s \in \mathbb{C}$

If we take the time domain solution and take the Laplace transform we conclude: $\mathcal{L}\{e^{At}\} = (sI - A)^{-1} \in \mathbb{C}^{n \times n}$

↳ examples in slides

$$\left(y(t) = C \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau + Du(t) = (K * u)(t), \quad K(t) = Ce^{At}B + D\delta u(t) \right)$$

Transfer function (TF)

If we insert ZSR ($x_0 = 0 \Rightarrow X(s) = (sI - A)^{-1} B U(s)$) in the formula for $Y(s)$, we get the transfer function $G(s)$:

$$\boxed{\begin{aligned} Y(s) &= C(sI - A)^{-1} B U(s) + D U(s) \\ &= \underbrace{(C(sI - A)^{-1} B + D)}_{:= G(s) \in \mathbb{C}^{p \times m}} U(s) = G(s) U(s) \end{aligned}}$$

Equivalent systems have the same transfer function.

System is called: - SISO if $m = p = 1$ $(B, C \in \mathbb{R}^n, D \in \mathbb{R})$

- MIMO if m or $p > 1$

For SISO:

- System is described by a rational function.

$$G(s) = \frac{C \cdot \text{adj}(sI - A) \cdot B + D \cdot \det(sI - A)}{\underbrace{\det(sI - A)}_{\text{characteristic polynomial!}}} = \frac{(s - z_1)(s - z_2) \dots (s - z_k)}{\underbrace{(s - p_1)(s - p_2) \dots (s - p_n)}_{chp(\lambda) \text{ only if no pole-zero cancellation}}}$$

system zeros

system poles

- TF is called proper if numerator degree $\leq$ denominator degree

\# poles $\leq$ \# zeros

↳ SISO TF arising from LTI systems are always proper

- TF is called strictly proper if numerator degree $<$ denominator degree

\# poles $<$ \# zeros

↳ SISO TF arising from LTI systems are strictly proper iff $D = 0$.
(i.e. input affects output only through system dynamics)

From now on, assume SISO, strictly proper TFs.

mon → 5.18

TF & Stability: Provided there are no pole zero cancellations

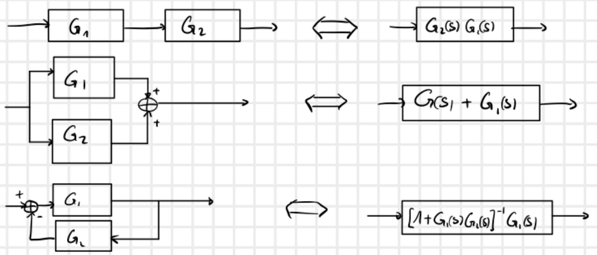
If poles are distinct, the system is:

- Asymptotically stable iff $Re[p_i] < 0, \forall i$
- Stable iff $Re[p_i] \leq 0, \forall i$
- Unstable iff $\exists i: Re[p_i] > 0$

If poles are repeated, the system is:

- Asymptotically stable iff $Re[p_i] < 0, \forall i$
- Unstable iff $\exists i: Re[p_i] > 0$
- If $\forall i\ Re[p_i] \leq 0$ and $\exists i\ Re[p_i] = 0$, the system may be stable or unstable, depending on partial fraction expansion. (depends on e-Vectors)

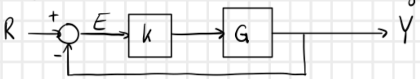
Cascadation of systems



↳ example: see slides

Closed loop system

"where you want to be" "where you are"



$R(s)$: Reference signal

$E(s) = R(s) - Y(s)$: Error Signal

G : "Plant" ~ system of interest

$$\boxed{E(s) = \frac{1}{1 + k\,G(s)} R(s) = H(s) R(s) = \frac{1}{F(s)} R(s)}$$

Usually, we want the error to decay, i.e.: $e(t) \to 0$ as $t \to \infty$

In other words, we want $H(s)$ to be asymptotically stable.

If we want an input output relation, we can simply use:

$$Y(s) = \frac{k\,G(s)}{1 + k\,G(s)} R(s)$$

Frequency response

We are interested in analysing the response of a system to sinusoidal input, with different frequencies. This is important bcs:

- Sinusoidal are common inputs
- Related to any other input with Fourier transform
- Frequency response, gives us some insight about the system behavior, e.g. will the system be stable in closed loop.

Frequency response can be summarized graphically with Bode- and Nyquist-plots.

$$G(s_0) = \frac{(s_0 - z_1)\cdots(s_0 - z_k)}{(s_0 - p_1)\cdots(s_0 - p_n)} = \frac{|s_0 - z_1| e^{j\angle(s_0 - z_1)} \cdots |s_0 - z_k| e^{j\angle(s - z_k)}}{|s_0 - p_1| e^{j\angle(s_0 - p_1)} \cdots |s_0 - p_n| e^{j\angle(s_0 - p_n)}}$$

$$= \frac{|s_0 - z_1| \cdots |s_0 - z_k|}{|s_0 - p_1| \cdots |s_0 - p_n|} e^{j[\angle(s_0 - z_1) + \dots + \angle(s_0 - z_k) - \angle(s_0 - p_1) - \dots - \angle(s_0 - p_n)]} = |G(s_0)| e^{j\angle G(s_0)}$$

$$\boxed{|G(s_0)| = \sqrt{Re(G(s_0)) + Im(G(s_0))}}, \quad \boxed{\angle G(s_0) = \tan^{-1}\left(\frac{Im[G(s_0)]}{Re[G(s_0)]}\right)}$$

Principle of the argument: (this is always the case!)

Assume that the curve D (arbitrary parametrisation of s_0 moving in $\mathbb{C}$) is traversed in clockwise direction and does not pass through any poles or zeros of $G(s)$. Let:

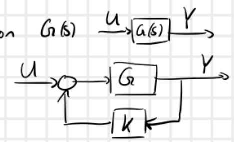
- N = Number of times L (curve drawn by $G(s_0)$ as s_0 traverses D) encircles (0,0) in the clockwise direction (clockwise → +1, anticlockwise → −1)
- Z = Number of zeros of $G(s)$ encircled by D
- P = Number of poles of $G(s)$ encircled by D

Then $\boxed{N = Z - P}$

Closed loop system II

open loop transfer function $G(s)$

closed loop $\frac{G(s)}{1 + k\,G(s)}$:



Find K s.t. closed loop stable?

p: number of open loop unstable poles in $G(s)$

z: number of closed — " — in $\frac{G(s)}{1 + k\,G(s)}$

N: number of encirclement of $(-\frac{1}{k}, 0)$

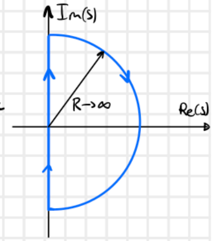
$Z = P + N$

? $G(s)$ Nyquist plot

stable $(z=0) \iff P = -N$

Nyquist diagram

We will now use the principle of the argument to argue on stability of the system. Since, for stability, we are only interested in the real part of the poles, we choose a litteral D as a curve for s_0. This results in an L curve known as Nyquist diagram.



We only need to analyse the top half of the curve since the bottom is only the complex conjugate.

We plot $Re[G(s)]$ against $Im[G(s)]$ following D.

Derive Nyquist plot from Bode plot (Moodle posted)

- If the phase of $G(j\omega)$ is 180^o then you can mark a point at $(-|G(j\omega)|, 0)$.
- If the phase of $G(j\omega)$ is 135^o then you can mark a point at $(-|G(j\omega)|/\sqrt{2}, |G(j\omega)|/\sqrt{2})$.
- If the phase of $G(j\omega)$ is 90^o then you can mark a point at $(0, |G(j\omega)|)$.
- If the phase of $G(j\omega)$ is 45^o then you can mark a point at $(|G(j\omega)|/\sqrt{2}, |G(j\omega)|/\sqrt{2})$.
- If the phase of $G(j\omega)$ is 0^o or 360^o then you can mark a point at $(|G(j\omega)|, 0)$.
- If the phase of $G(j\omega)$ is 315^o then you can mark a point at $(|G(j\omega)|/\sqrt{2}, -|G(j\omega)|/\sqrt{2})$.
- If the phase of $G(j\omega)$ is 270^o then you can mark a point at $(0, -|G(j\omega)|)$.
- If the phase of $G(j\omega)$ is 225^o then you can mark a point at $(-|G(j\omega)|/\sqrt{2}, -|G(j\omega)|/\sqrt{2})$.

Draw these points as much as possible, and then trace through all these points to sketch the Nyquist plot.

Nyquist stability creterion

Consider the following quantities from the curve L in the Nyquist plot of $G(s)$. Let:

- N = Number of times L encircles $(\frac{-1}{k}, 0)$ in the clockwise direction.
- Z = Number of poles of $H(s)$ with positive real part.
- P = Number of poles of $G(s)$ or zeros of $H(s)$ with positive R-part

↳ The closed loop system is stable ($Z=0$) iff

$$\boxed{N = -P}$$

From Nyquist From G(s)

- So given P we know in what region $-\frac{1}{k}$ should be. Then, we can use this to determine the intervall for k. (S7.A1.5)

- Another way to show stability is to ensure that the CL TF is asymptotically stable, i.e. $\frac{kG(s)}{1+kG(s)}$ has poles with negative real part or easier $\frac{1+k\,G(s)}{k\,G(s)}$ has zeros with negative real part.

Compute the maximum magnitude of $\|G(j\omega)\|$:

When we have 2 poles: $G(s) = \frac{1}{s^2 + a_0 s + a_1}$, we can compare the TF to $\frac{K'\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

then max at $\omega = \omega_0\sqrt{1 - \zeta}$

Second order filter → 5.47

Bode Plots

Bode plots: Pair of plots with the $log(\omega)$ (in rad/sec) along the x-axis with the y-axes plotting $20\log(|G(j\omega)|)$(in dB) and $\angle G(j\omega)$ (in degrees).

Useful hints: The phase ϕ and the magnitude change per decade $\triangle M$ of standard elements $G(j\omega)$ is given depending on the frequency ω, where $\alpha > 0$

	$G(j\omega)$	$\phi(\omega=0)$	$\phi(\omega=\infty)$	$\triangle M(\omega=0)$	$\triangle M(\omega=\infty)$
Integrator	$\frac{1}{s}$	$-90°$	$-90°$	-20dB	-20dB
Differentiator	s	$90°$	$90°$	20dB	20dB
Stable Pole	$\frac{1}{(s+\alpha)}$	$0°$	$-90°$	0dB	-20dB
$\cancel{(s-\alpha)}$ $(\alpha-s)$ ⚠ Unstable Pole	$\frac{1}{(\alpha-s)}$	$0°$	$90°$	0dB	-20dB
Stable Zero	$(s+\alpha)$	$0°$	$90°$	0dB	20dB
Unstable Zero	$(\alpha-s)$	$0°$	$-90°$	0dB	20dB

Note: Multiplication by a scalar shifts the entire magnitude plot vertically by a constant offset. Multiplication by a positive scalar has no effect on the phase plot but a negative scalar shifts the phase by 180°.

If the magnitude is decaying from the start, we can determine the zero-frequency gain by extrapolating: we know the gain to be x at some frequency y, then we insert y in $G(y) = A \cdot \frac{(y-0,1)\dots}{(y+10)\dots} = x$ and obtain A from this.

From EC)

1. Bring transfer function in the following form.

$$H(j\omega) = A \cdot \left(\frac{\left(1+j\frac{\omega}{\omega_{z1}}\right)\cdot \ldots \cdot\left(1+j\frac{\omega}{\omega_{zk}}\right)}{\left(1+j\frac{\omega}{\omega_{p1}}\right)\cdot \ldots \cdot\left(1+j\frac{\omega}{\omega_{pl}}\right)}\right) = A\left(\frac{\left(1+\frac{s}{\omega_{z1}}\right)\cdot \ldots \cdot\left(1+\frac{s}{\omega_{zk}}\right)}{\left(1+\frac{s}{\omega_{p1}}\right)\cdot \ldots \cdot\left(1+\frac{s}{\omega_{pl}}\right)}\right)$$

2. Identify and mark all zeros $|\omega_z|$ and poles $|\omega_p|$ (written as $a \cdot 10^b$)
3. Start Amplitude @ $20 \cdot log_{10}(|A|)$.
Start phase according to poles and zeros in 0 as well as the phase of A
4. From left to right use this table

	Formula	Amplitude	Phase (over 2 decades)
LHP Zero,	$1+\frac{j\omega}{\omega_z}$	+20dB/Decade	$+90°$ (@ $\omega_z + 45°$)
RHP Zero,	$1-\frac{j\omega}{\omega_z}$	+20dB/Decade	$-90°$ (@ $\omega_z - 45°$)
LHP Pole,	$\frac{1}{1+\frac{j\omega}{\omega_p}}$	−20dB/Decade	$-90°$ (@ $\omega_p - 45°$)
RHP Pole,	$\frac{1}{1-\frac{j\omega}{\omega_p}}$	−20dB/Decade	$+90°$ (@ $\omega_p + 45°$)
Zero @ $\omega_z = 0$ Differentiator	$\frac{j\omega}{\omega_0}$	Start with +20dB/Dec	Start @ $+90°$
Pole @ $\omega_p = 0$ Integrator	$\frac{\omega_0}{j\omega}$	Start with -20dB/Dec	Start @ $-90°$
Constant A	$\lvert A\rvert e^{i\varphi}$	$20 \cdot log_{10}(\lvert A\rvert)$.	Start @ $\angle(A) = \varphi$

Amplitude: $y = 20 \cdot \log_{10}(|H(j\omega)|) = 20 \cdot \log_{10}(|H(j10^x)|)$

Phase: $y = arctan\left(\frac{Im(H(j10^x))}{Re(H(j10^x))}\right) \cdot \frac{180}{\pi}$

For a transfer function H which is the product of 2 other transferfunctions $H = H_1 \cdot H_2$ we have:

$$20\log_{10}(H_1 \cdot H_2)\,dB = (20\log_{10}(H_1) + 20\log_{10}(H_2))dB$$

Therefore we can just add the magnitude and phases of each plot of H_1&H_2 and get H.

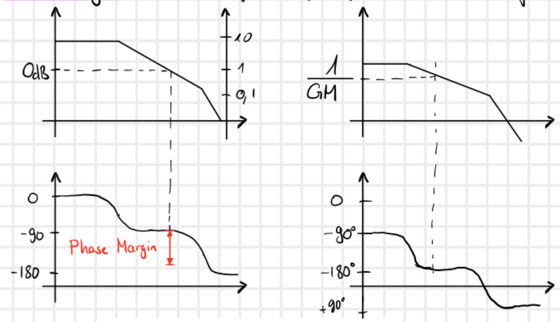
$$\boxed{G(s) = \frac{G_1(s)}{G_2(s)} \quad \Rightarrow \quad \begin{array}{l} |G|_{dB} = |G_1|_{dB} - |G_2|_{dB} \\ \angle G = \angle G_1 - \angle G_2 \end{array}}$$

$$\boxed{G(s) = G_1(s) \cdot G_2(s) \quad \Rightarrow \quad \begin{array}{l} |G|_{dB} = |G_1|_{dB} + |G_2|_{dB} \\ \angle G = \angle G_1 + \angle G_2 \end{array}}$$

Assuming the open loop system $G(s)$ is stable, we have:

Gain margin: The factor $\frac{1}{|KG(j\omega_u)|} \geq 1$ where $\angle G(j\omega_u) = -180°$

Phase margin: Phase: $\angle G(j\omega_c) + 180°$ (>0) where $|G(j\omega_c)| = 1 = 0dB$



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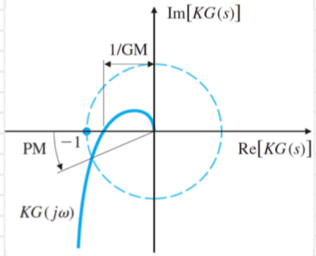


P&G margin s.t. -1 isn't encircled in clockwise direction

Steady-state Response to sinusoidal excitation

Input: sinusoidal excitation with amplitude K and frequency ω_0:

$y(t) = K \cdot |G(j\omega_0)| \sin(\omega t + \angle G(j\omega_0))$

Rmk: - if the system is not stable, there is no steady-state response

Step-response & Impulse response

In order to calculate the step/impulse response, we go in the Laplace domain and calculate:

$Y(s) = G(s) \cdot U(s)$ with $U(s)$ beeing the laplace trafo of the input

$\downarrow \mathcal{L}^{-1}$: then tranform the soln in time domain using partial fraction decomp.

$y(t)$

step-input: $u_s(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases}$ ○—● $U_s(s) = \frac{1}{s}$

impulse-input: $u_i(t) = \delta(t)$ ○—● $U_i(s) = 1$

For a 2nd order system, the impulse response of a system with $G(s) = \frac{\omega_0}{(s-\sigma)^2 + \omega_0^2}$ is an oscillating function with frequency $\frac{\omega_0}{2\pi}$ and exponential decay with time $\propto \sigma$. $e^{-\sigma t}$

Other signal-response: try to construct the given signal out of known signal and use Linearity + shift property to compute the whole response

Bode stability criterion:

Assume open loop transfer function $G(s)$ is asymptotically stable and its magnitude and phase Bode plots are monotone decreasing.
Then the closed loop system is stable if and only if $|KG(j\omega)| < 1$ at the frequency where $\angle G(j\omega) = -180°$.

Small gain Theorem: Assume the openloop transferfunction $G(s)$ is asymptotically stable and its magnitude and Phase are monotonically decreasing, if

$|KG(j\omega)| < 1 \quad \forall \omega$

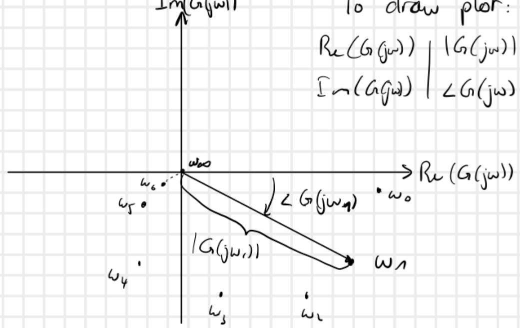
then the closed loop system is stable.

2nd Order Systems

For TF of the form $G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ see slide 5.48

If not exactly in this form, to argue that $|G(j\omega)|$ has a maximum, take the derivative of the square of the norm of the denominator and set it to 0.

- Frequency domain
  - sinusoidal input: $u(t) = \sin(\omega t) \cdot 1$
    ω: frequency
    $y(t) = |G(j\omega)| \sin(\omega t + \angle G(j\omega))$
    $G(s)$ transfer f. $s = j\omega$
    magnitude: $|G(j\omega)| = \sqrt{Re(G(j\omega))^2 + Im(G(j\omega))^2}$
    phase: $\angle G(j\omega) = \tan^{-1}\left(\frac{Im(G(j\omega))}{Re(G(j\omega))}\right)$
  - Nyquist plot: $G(j\omega) = |G(j\omega)| e^{j\angle G(j\omega)}$



To draw plot:

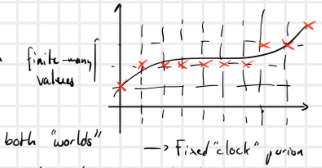
$Re(G(j\omega))$	$\lvert G(j\omega)\rvert$
$Im(G(j\omega))$	$\angle G(j\omega)$

Discrete time LTI Systems

Sampled data systems

Computers operate on bits streams which implies Value and time quantization.

Digital computers have to interact with analog environment.



This requires transformation between both "worlds"

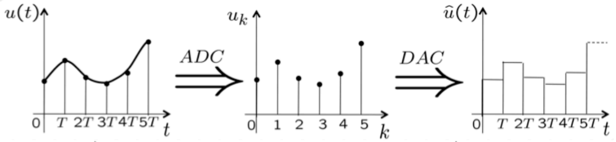
→ Analog to digital conversion (A/D or ADC)

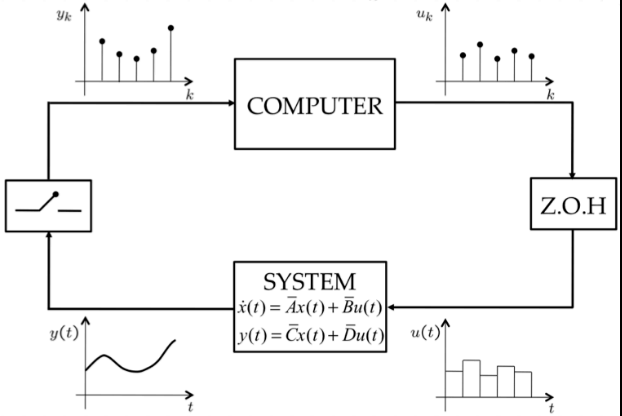
← Digital to analog (D/A or DAC)

Usually value quantization is good enough so let's focus on time.

Assume: "ADC": sample every T-seconds

"DAC": zero order hold





Sampled Data Linear System

Starting from a continuous LTI system, if we sample at every T, we get:

$$\boxed{\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned}} \quad \text{with} \quad A = e^{\bar{A}T},\ B = \int_0^T e^{\bar{A}(T-\tau)}\bar{B}\,d\tau, \quad C = \bar{C},\ D = \bar{D}$$

$$\left(\int_0^T \tau e^{\tau}\, d\tau = Te^T - e^T + 1\right)$$

Discrete time linear systems

General formulation: $x_{k+1} = Ax_k + Bu_k$ $\quad x_k \in \mathbb{R}^n,\ u_k \in \mathbb{R}^m,\ y_k \in \mathbb{R}^p$

$y_k = Cx_k + Du_k$ $\quad A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$

Solution: given $\hat{x}_0 \in \mathbb{R}^n$ and $u_k \in \mathbb{R}^m$, $k = 0,1,\dots,N-1$, the solution consists of two sequences $x_k \in \mathbb{R}^n$ and $y_k \in \mathbb{R}^p$, $k = 0,1,\dots,N$ s.t. the eqn from above hold $\forall k$.

$$\boxed{\begin{aligned} x_k &= \underbrace{A^k \hat{x}_0}_{ZIT} + \underbrace{\sum_{i=0}^{k-1} A^{k-i-1} B u_i}_{ZST} \\ y_k &= Cx_k + Du_k \end{aligned}}$$

A^k can be computed with E-W decomposition: if A is diagonalizable:

$$A = W\Lambda W^{-1} \Rightarrow A^k = W\Lambda^k W^{-1} \qquad \Lambda^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}$$

Stability

If A diagonalizable, A^k is a linear combination of λ_i^k.

> Thm: System with diagonalizable A is:
> - Stable iff $|\lambda_i| \leq 1\ \forall i$
> - Asymptotically stable iff $\forall i\ |\lambda_i| < 1$
> - Unstable iff $\exists i: |\lambda_i| > 1$

It can be shown: if $\bar{A}$ is diagonalizable and $Re[\lambda_i] < 0\ \forall i$ then $A = e^{\bar{A}T}$ is also diagonalizable and $|\lambda_i| < 1, \forall i$.

> Thm: For a non-diagonalizable Matrix: The system is
> - Asymptotically stable iff $\forall i\ |\lambda_i| < 1$
> - Unstable iff $\exists i: |\lambda_i| > 1$
> - Could be stable or unstable if $\exists i: |\lambda_i| = 1$

Deadbeat response

Assume all e-values of A are zero (dead beat control): $\lambda_1 = \dots = \lambda_n = 0$

Then $A^N = 0$ for some $N \leq n$ (nilpotent matrix)

From this follows for ZIT: $x_k = A^k x_0 = 0\ \ \forall k \geq N$, i.e. ZIT gets to 0 in finite time and stays there.

This never happens with continuous time systems. (it falls asymptotically, but never 0 if $x_0 \neq 0$)

Coordinate change (analog to continuous)

Assume $\hat{x}_k = Tx_k$ for some invertible $T \in \mathbb{R}^{n\times n}$.

$$\begin{aligned} \hat{x}_{k+1} &= \hat{A}x_k + \hat{B}u_k \\ y_k &= \hat{C}x_k + \hat{D}u_k \end{aligned} \quad \text{where} \quad \begin{aligned} \hat{A} &= TAT^{-1}, & \hat{B} &= TB \\ \hat{C} &= CT^{-1}, & \hat{D} &= D \end{aligned}$$

Energy and Power:

Consider "energy-like" function: $V(x) = \frac{1}{2}x^T Q x \qquad Q = Q^T > 0$

The Power (change of energy in time) is given by:

$$P = V(x_{k+1}) - V(x_k) = \frac{1}{2}x_k^T(A^TQA)x_k + \frac{1}{2}u_k^TB^TQBu_k + \frac{1}{2}u_k^TB^TQAx_k + \frac{1}{2}x_k^TA^TQBu_k$$

$$\overset{u_k = 0}{=} \frac{1}{2}x_k^T(A^TQA - Q)x_k = -\frac{1}{2}x_k^T R x_k$$

> Thm: $|\lambda_i| < 1\ \ \forall i = 1,2,\dots,n$ iff $\forall R = R^T > 0$ the equation $(A^TQA - Q) = -R$ has a unique solution with $Q = Q^T > 0$

~ if we can find a Q s.t. $\underbrace{\text{Power}}_{= P = -\frac{1}{2}x_k^TRx_k}$ is negative $\forall x, t$ then stable. $\quad R = R^T > 0$

Controllability

System is controllable if we can steer it from any $\hat{x}_0 \in \mathbb{R}^n$ to any final condition $\hat{x}_N \in \mathbb{R}^n$ using appropriate sequence of input u_k, $k = 0,1,\dots,N-1$

We must assume that we have enough time, i.e. $N \geq n$

$$P = [B \quad AB \quad A^2B \ \dots \ A^{n-1}B]$$

> Thm: The system is controllable iff P has rank n

Rmk: - The readable set from $\hat{x}_0 = 0$ is Range[P]

- Minimum energy control, we want to solve $\hat{x}_n - A^n\hat{x}_0 = P \cdot U$

If $m = 1$, $P \in \mathbb{R}^{n\times m}$, Rank[P] $= n \Leftrightarrow P^{-1}$ Exists

$\boxed{U = P^{-1}[\hat{x}_n - A^n x_0]}$ ← unique solution

If $m > 1$ or $N > n$, P is not square

$\boxed{U = \underbrace{P^T(PP^T)^{-1}}_{P^{\dagger} \sim \text{pseudo inverse}}(\hat{x}_n - A\hat{x}_0)}$ ← one of many solutions
But among all solutions that solves the eqn, this one has the smallest $\|U\|$
⇒ Minimum energy control

- If the system is uncontrollable, we have a pole zero cancellation in the TF.

Observability

System is observable if we can infer the state evolution $x_k, k=0,1,\dots,N$ by observing $u_k, y_k, k=0,1,\dots,N$

Assuming $N \ge n-1$: $Q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

$\boxed{\text{Thm: The system is observable iff } Q \text{ has rank } n.}$

Error dynamics

$e_k = \hat{x}_k - x_k$

Always try to express $e_{k+1}(e_k)$. So try to equate

$\hat{x}_{k+1} - x_{k+1} = \dots$ to some expression in function of e_k

z-Transform

Let $f_k = 0\ \forall k<0$, and f_k is such that the sum converges:

$f: \mathbb{N} \to \mathbb{R}$ $\quad f_k \underset{\mathcal{Z}^{-1}}{\overset{\mathcal{Z}}{\rightleftarrows}} F(z) \quad F: \mathbb{C} \to \mathbb{C}$

$$\boxed{F(z) = \mathcal{Z}\{f_k\} = \sum_{k=0}^{\infty} f_k z^{-k}}$$

$z \in \mathbb{C}$ can be thought as unit time delay.

Properties: see 6.20

Or compute directly S7A3.2

Transfer Function

Assuming $x_0 = 0$, and taking Z-transform of all signals

$x_{k+1} = Ax_k + Bu_k \Rightarrow zX(z) = AX(z) + BU(z)$

$y_k = Cx_k + Du_k \Rightarrow Y(z) = CX(z) + DU(z)$

$$\boxed{Y(z) = \underbrace{\left[C(zI-A)^{-1}B + D\right]}_{\text{Transfer function} := G(z)} U(z)}$$

Rmk: - TF is a rational function of z

- System asymptotically stable $\iff$ Poles of $G(z)$ have magnitude <1
- If the system is uncontrollable or unobservable there is pole zero cancellation
- The number of poles is the same as the dimension n (when no pole-zero cancellation occurs)

$\Rightarrow$ We know that if # poles $\neq n$, we have pole-zero cancellation and an unobservable/uncontrollable system.

- The poles are the same as the E-values of A!
- If an unstable pole is cancelled by an observer (p-z cancel.) and that the resulting system is stable, we must be very carefull since if p and z do not exactly match (which is most often the case in practice) the system becomes unstable

Simulation (~ link between continuous and discrete systems)

A simulation is a numerical solution in the computer.

- We use Numerical approximation to approximate the solution with a sequence $\{x_k\}_{k=0}^{N}$.
  - Divide $[0,T]$ in N equal subintervals
  - Let $\delta = \frac{T}{N}$ be a simulation step
  - $x((k+1)\delta) \approx x(k\delta) + \delta\dot{x}(k\delta) = x(k\delta) + \delta(Ax(k\delta) + Bx(k\delta))$.

    $x_{k+1} = (I + A\delta)x_k + \delta B u_k$
- Zero input response

Consider autonomous system with solution $x(t) = \Phi(t)x_0 = e^{At}x_0$

$x((k+1)\delta) = e^{A((k+1)\delta - k\delta)} x(k\delta) = e^{A\delta} x(k\delta) = (I + A\delta + \frac{A^2\delta^2}{2} + \dots)x(k\delta)$

1st order: $\boxed{x((k+1)\delta) \approx (I + A\delta)x_k}$, $x_k \approx (I + A\delta)^k x_0$

- Stability of numerical approximation

~ How small should the steps be?

Hard question to answer but we would like that if system asympt. stable then numerical approx also goes to zero.

↳ Assume A diagonalizable $A = W\Lambda W^{-1}$

Then $x_k = (I + A\delta)^k x_0 = (WW^{-1} + W\Lambda W^{-1}\delta)x_0 = W(I + \Lambda\delta)^k W^{-1} x_0$

So discrete time system is asymptotically stable iff

$x_k \to 0\ \forall x_0 \in \mathbb{R}^n \iff |1 + \lambda_i\delta| < 1\ \forall i = 1,\dots,n$

And if λ_i are real and negative, we have: $\boxed{\delta < \frac{2}{\max_{i=1,\dots,n} |\lambda_i|}}$

This method is the Euler Method: $x_{k+1} = x_k + \delta\dot{x}_k$

$= x_k + \delta A x_k + \delta B u_k$

$= (\delta A + I)x_k + \delta B u_k$

Final value theorem for z-transform

If all poles of $(z-1)F(z)$ are inside the unit circle:

$$\lim_{k\to\infty} f(k) = \lim_{z\to 1} (z-1)F(z)$$

Periodic system response

If we want a system to have a periodic response to a periodic input with period $T=k$, we must set $y_0 = y_k$, if the output is autonomous, i.e. $y(x)$, then x must also be periodic and we can set $x_0 = x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} B u_i$ and solve for the corresponding initial condition.

Linearized discretization

If we linearize a discrete-time system (e.g. around $x=0$ by assuming $\|x_k\|$ is very small and $x_i^3 = 0$), we cannot tell, distinguish if the equilibrium around which we linearized the system is globally or only locally stable. This is because the linearized system will always have only one equilibrium and thus be GAS (or unstable). But the original system may have many equilibria and thus not GAS. We lost information through the linearization.

Non-linear systems

~ Almost all systems are non linear (globally)!
Almost all systems are linear (locally)!

We return to the general systems:

$\dot{x}(t) = f(x(t), u(t))$

$y(t) = h(x(t), u(t))$

But we concentrate on autonomous systems i.e.: $\dot{x}(t) = f(x(t))$ and we further assume that f is Lipschitz ($\Rightarrow$ existance & uniqueness)

Invariant sets: $\boxed{\text{A set of states } S \subseteq \mathbb{R}^n \text{ is called invariant if } \forall x_0 \in S, \forall t \geq 0,\ x(t) \in S}$

Equilibrium: $\boxed{\text{A state } \hat{x} \in \mathbb{R}^n \text{ is called an equilibrium if } f(\hat{x}) = 0}$

Rmk: – In general $f(\hat{x})$ is a set of n nonlinear equation, n unknowns and can have:
- no soln
- 1 soln
- finitely many
- countably many
- uncountably many (subspace)

} ← linear systems

– Linear systems have a linear subspace of equilibria.
↳ the Null space of A!

– Nonlinear systems can have many isolated equilibria.
they have to solve the equation $f(\hat{x}) = 0$

– We can move the equilibrium to 0, with a nonlinear (affine) change of coordinates. $w(t) = x(t) - \hat{x} \in \mathbb{R}^n$

$$\left.\begin{aligned} \dot{x}(t) &= f(x(t)) \\ f(\hat{x}) &= 0 \end{aligned}\right\} \underset{x(t) = w(t) + \hat{x}}{\overset{W(t) = x(t) - \hat{x}}{\rightleftarrows}} \left\{\begin{aligned} \dot{w}(t) &= \frac{d}{dt}(x(t) - \hat{x}) = \dot{x}(t) - \cancelto{0}{\frac{d}{dt}\hat{x}} = \\ &= \dot{x}(t) = f(x(t)) = f(w(t) + \hat{x}) = \hat{f}(w(t)) \end{aligned}\right.$$

The system in the new coordinates has an equilibrium at $\hat{w} = 0$

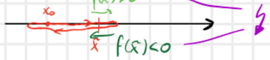
$\hat{f}(0) = f(0 + \hat{x}) = f(\hat{x}) = 0$

Limit cycles: $(\mathbb{R}^n, n \geq 2)$

$\boxed{\text{A solution } x(t) \text{ is called a periodic orbit if } \exists T > 0,\ \forall t \geq 0,\ x(t+T) = x(t)}$

$\boxed{\text{The Period is the smallest } T \text{ such that } x(t) = x(t+T)\ \forall t \geq 0}$

Rmk: – Equilibria define trivial periodic orbits with $T = 0$
– Limit cycle: Non-trivial periodic orbit.
– Linear systems exhibit either: – Trivial periodic orbits
– Subspaces of periodic orbits
– Scalar systems $x(t) \in \mathbb{R}$ have no limit cycles



van der Pol, Strange attractors, chaotic attractor, Lorenz attractor → slides

Stability

$\boxed{\begin{array}{l}\text{An equilibrium } \hat{x} \text{ is called stable if } \forall \varepsilon > 0 \text{ there exists } \delta > 0 \\ \text{such that: } \|x_0 - \hat{x}\| < \delta \Rightarrow \|x(t) - \hat{x}\| < \varepsilon \quad \forall t \geq 0. \\ \text{Otherwise equilibrium unstable}\end{array}}$

~ if we start close, we stay close

LAS

$\boxed{\begin{array}{l}\text{An equilibrium } \hat{x} \text{ is called locally asymptotically stable if} \\ \text{it is stable and } \exists M > 0 : \|x_0 - \hat{x}\| < M \Rightarrow \lim_{t\to\infty} x(t) = \hat{x} \\ \text{It is called globally asymptotically stable (GAS) if this holds for any } M > 0\end{array}}$

$\boxed{\text{The set of } x_0 \text{ such that } \lim_{t\to\infty} x(t) = \hat{x} \text{ is called the domain of attraction of } \hat{x}.}$

Rmk: – Non linear: GAS $\subsetneq$ LAS $\subsetneq$ Stable
– Linear: $\underbrace{\text{GAS} = \text{LAS}}_{Re\{\lambda_i\} < 0} \subsetneq \underbrace{\text{Stable}}_{Re\{\lambda_i\} \leq 0}$

Stability by Linearization

Easy way to study e.g. stability of non linear system by approximating it with a linear system.

We do this by doing a taylor expansion about $\hat{x}$:

$$f(x) = f(\hat{x}) + A(x - \hat{x}) + \text{higher order terms in } (x - \hat{x}) \approx \cancelto{0}{f(\hat{x})} + A(x - \hat{x}) + \cancel{\text{higher order}}$$

equilibrium

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $f(x) = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix}$, $A = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n}(\hat{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\hat{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\hat{x}) \end{bmatrix}}_{\text{Jacobian computed @ } \hat{x}}$

Then, consider $\delta x(t) = x(t) - \hat{x}$. When we are close to the equilibrium point, δx is small and $\frac{d\delta x(t)}{dt} \approx A\delta x(t)$

↳ So close to the equilibrium, nonlinear systems behave more or less linearly. We can then examine stability of the equilibrium by looking at the eigenvalues of A.

$\boxed{\begin{array}{l}\text{Thm 7.1 Lyapunov first method: The equilibrium } \hat{x} \text{ is} \\ \text{1. LAS if the eigenvalues of } A \text{ have negative real part} \\ \text{2. Unstable if at least one eigenvalue of } A \text{ has positive real part.}\end{array}}$

Disadvantages of method: – no info about the domain of attraction
– Inconclusive if $\exists i: Re[\lambda_i] = 0$

example of linearisation: pendulum, see slides

Lyapunov functions

We remember that in linear systems, stability is characterized in two ways: – E-values of A
– Existance of a decreasing energy-like function

The first way is reused with linearisation.
The second we will slightly change but keep the same principle.

$\boxed{\begin{array}{l}\text{Thm 7.2 Lyapunov second method: Assume there exists an} \\ \text{open set } S \subseteq \mathbb{R}^n \text{ with } \hat{x} \in S \text{ and differentiable function} \\ V(\cdot): \mathbb{R}^n \to \mathbb{R}, \text{ such that:} \\ \quad 1.\ V(\hat{x}) = 0 \\ \quad 2.\ V(x) > 0 \quad \forall x \in S \text{ with } x \neq \hat{x} \\ \quad 3.\ \frac{d}{dt} V(x(t)) \leq 0,\ \forall x \in S \\ \text{Then the equilibrium } \hat{x} \text{ is stable}\end{array}}$

Rmk: – $V(x)$ is known as Lyapunov function
– Derivatives along the trajectories known as Lie derivatives:

$$\frac{d}{dt} V(x(t)) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x(t)) \cdot \frac{d}{dt} x_i(t) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}(x(t)) f_i(x(t))$$

$$= \vec{\nabla} V(x(t)) \quad \vec{f}(x(t)) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \cdots & \frac{\partial V}{\partial x_n} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix}$$

↳ proof and example pendulum: see slides $V(x) \sim \frac{1}{2}x_2^2 + 1 - \cos(x_1)$
By changing $V(x)$ of pendulum slightly, it can be fitted to many other systems.
↳ If one wants to check this for linear systems: $\frac{dV}{dt} = \dot{x}^T Q x + x^T Q \dot{x}$

$\boxed{\begin{array}{l}\text{Thm 7.3 Assume there exists an open set } S \subseteq \mathbb{R}^n \text{ with} \\ \hat{x} \in S \text{ and a differentiable function } V(\cdot): \mathbb{R}^n \to \mathbb{R}, \text{ such that:} \\ \quad 1.\ V(\hat{x}) = 0 \\ \quad 2.\ V(x) > 0,\ \forall x \in S, \text{ with } x \neq \hat{x} \\ \quad 3.\ \frac{d}{dt} V(x(t)) < 0,\ \forall x \in S \text{ with } x \neq \hat{x} \\ \text{Then the equilibrium } \hat{x} \text{ is LAS}\end{array}}$

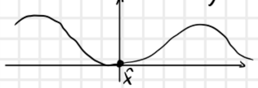
↳ This can help estimate the domain of attraction. If we can find $c > 0$ such that $\{x \in \mathbb{R}^n \mid V(x) \leq c\} \subseteq S$, then the trajectory that start in this set and close to $\hat{x}$ stay in the set and converge to $\hat{x}$, i.e. belong to the domain of attraction of $\hat{x}$

Thm 7.4: Assume there exists a differentiable function $V(\cdot): \mathbb{R}^n \to \mathbb{R}$ such that:

1. $V(\hat{x}) = 0$
2. $V(x) > 0, \forall x \neq \hat{x}$
3. $\|x\| \to \infty \Rightarrow V(x) \to \infty$
4. $\frac{d}{dt} V(x(t)) < 0, \forall x \neq \hat{x}$

Then the equilibrium $\hat{x}$ is GAS

Rmk: - Condition 3 must hold as x increases in any direction and ensures that it is radially unbounded, i.e. this doesn't happen:



- Example → see slides 7.31

Thm 7.5 La Salle's Theorem: Assume there exists a compact invariant set $S \subseteq \mathbb{R}^n$ and a differentiable function $V(\cdot): \mathbb{R}^n \to \mathbb{R}$ such that

$$\dot{V}(x) = \nabla V(x) f(x) \leq 0 \qquad \forall x \in S$$

Let M be the largest invariant set contained in the set

$$\bar{S} = \{x \in S \mid \nabla V(x) f(x) = 0\} \subseteq \mathbb{R}^n$$

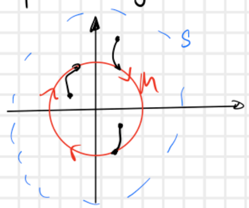
Then all trajectories starting in S tend to M as $t \to \infty$

Rmk: - Compact means bounded and closed $\sim [a,b]$
- If $\hat{x}$ is the only invariant set in

$$\{x \in S \mid \nabla V(x) f(x) = 0\}$$

then all trajectories starting in S tend to it.
- see s.8 11.3 for an example.
  s8 12 → stable limit cycle

example: periodic cycle: $\exists T$ s.t. $\forall t > 0 \quad x(t+T) = x(t)$



$\exists T$ s.t. $\forall t > 0$
$x(t+T) = x(t)$

$V(x) = \mathrm{dist}(x, \circlearrowleft)$

Lyapunov Methods (aus dem PVK)

i) Lyapunov first/indirect method:

Linearise $\dot{x} = f(x)$ around equil. $\hat{x}$:

$$\dot{x} = f(x) \simeq \underset{\substack{\| \\ 0}}{f(\hat{x})} + \underbrace{J_{f|\hat{x}}}_{=A} (x - \hat{x}) \qquad \text{(1st order Taylor approx) local}$$

$\hat{x}$ equil. → $f(\hat{x}) = 0$

$$J_{f|\hat{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

$\hat{x}$ locally asymptotically stable if $\mathrm{Re}(\lambda_i) < 0 \ \forall i$

$\hat{x}$ unstable if $\exists i$: $\mathrm{Re}(\lambda_i) > 0$

If $\mathrm{Re}(\lambda_i) = 0$ inconclusive

ii) Lyapunov second/direct method.

If $\exists$ a diff. function $V(x): \mathbb{R}^n \to \mathbb{R}$

① 1) open set $S \subseteq \mathbb{R}^n$ with $\hat{x} \in S$
2) $V(\hat{x}) = 0$
3) $V(x) > 0 \quad \forall x \in S, x \neq \hat{x}$
4) $\frac{dV}{dt}(t) = \frac{dV}{dx}^T \underbrace{\frac{dx}{dt}}_{= \dot{x} = f(x)} \leq 0 \quad \forall x \in S$ $\qquad \left(\frac{dV}{dx} = J_V = \nabla_x V\right)$

⇒ then $\hat{x}$ is stable

② 4') $\frac{dV(x)}{dt} < 0 \quad \forall x \in S, x \neq \hat{x}$

⇒ then $\hat{x}$ is asymptotically stable

③ 1) $S = \mathbb{R}^n$
4') the same: $\frac{dV(x)}{dt} < 0 \quad \forall x \in S, x \neq \hat{x}$
5) $V(x) \xrightarrow[\|x\| \to \infty]{} \infty$

⇒ then $\hat{x}$ is globally asymptotically stable

Invariant Set S: - $x(0) \in S \Rightarrow x(t) \in S, \forall t$
- La Salle's invariance principle

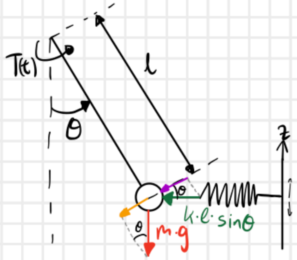
$\exists S \subseteq \mathbb{R}^n$ compact (closed and bounded) and S invariant and $\exists$ a diff. $V(x): \mathbb{R}^n \to \mathbb{R}$. $\frac{dV}{dx}^T f(x) \leq 0 \quad \forall x \in S$

M largest invariant set s.t. $M \subseteq \bar{S} = \{x \in S \mid \frac{dV}{dx}^T f(x) = 0\} \subseteq S$

$x(t) \xrightarrow[t \to \infty]{} M$ (if $x(0) \in S$)

$f(t)$	$\mathcal{L}(f)(s) = F(s)$
1	$\frac{1}{s}, \quad s > 0$
t^n	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}, \quad s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2} \quad s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2} \quad s > a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}} \quad s > a$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$tf(t)$	$-F'(s)$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$e^{at}f(t)$	$F(s-a)$

Example torque: We are interested in following θ. Spring is relaxed when $\theta = 0$, but moves freely along the z-Axis (always same lvl as m)



T(t): Input Torque

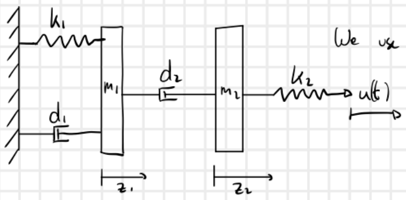
$$I\ddot{\theta} = T(t) - \underline{l \cdot \underline{mg \sin\theta}} - \underline{\underline{k l \cdot \sin\theta} \cdot \cos\theta}$$

$$x(t) = \begin{pmatrix} \theta \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

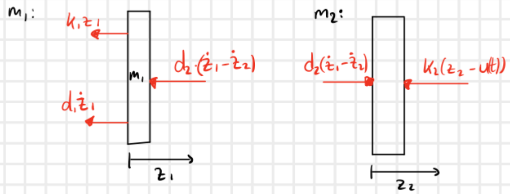
$$f(x(t)) = \dot{x}(t) = \begin{pmatrix} \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} x_2 \\ T(t) - lmg\sin(x_1) - kl\sin(x_1)\cos(x_1) \end{pmatrix}$$

Example Forces:

All positions are measured w.r.t. the equilibrium state where all forces are 0.



We use $y(t) = z_2(t)$ as output



$$m_1\ddot{z}_1 = d_2(\dot{z}_2 - \dot{z}_1) - k_1 z_1 - d_1\dot{z}_1 \qquad m_2\ddot{z}_2 = d_2(\dot{z}_1 - \dot{z}_2) - k_2(z_2 - u(t))$$

$$x(t) = \begin{bmatrix} z_1 \\ \dot{z}_1 \\ z_2 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$f(x) = \dot{x}(t) = \begin{bmatrix} \dot{z}_1 \\ \ddot{z}_1 \\ \dot{z}_2 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{d_2}{m_1}(x_4 - x_2) - \frac{k_1}{m_1}x_1 - \frac{d_1}{m_1}x_2 \\ x_4 \\ \frac{d_2}{m_2}(x_2 - x_4) - \frac{k_2}{m_2}(x_3 - u(t)) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{m_1} & -\frac{d_1+d_2}{m_1} & 0 & \frac{d_2}{m_1} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{d_2}{m_2} & -\frac{k_2}{m_2} & -\frac{d_2}{m_2} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_2}{m_2} \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 0 \quad 1 \quad 0) \; x(t)$$