Formula sheet Analysis III - Jean Mégret

1 Preliminaries

let $u = u(x,y,z)$

Gradient: $\nabla u = (u_x\ u_y\ u_z)^T$

Laplacian: $\Delta u = \nabla^2 u = u_{xx} + u_{yy} + u_{zz}$

Heat equation: $u_t = \Delta u$

Laplace equation: $\Delta u = 0$

Wave equation: $u_{tt} = c^2 \Delta u$

Burgers equation: $u_t = u u_x$

2 Types of conditions: Boundary $(0,t)$, Initial $(x_0, 0)$

Well posed problem: { 1- Existence: Prblm has a solution
2- Uniqueness: Prblm has only one soln.
3- Stability: Small change in eqn or data → small change in the solution

otherwise ill-posed

Strong solutions: if all the derivatives of the soln that are in the PDE exist and are continuous.

Weak solutions: Only valid in some domain.

Classification of PDEs

- Order: highest order partial derivative $u_{xyz} \to 3$
- Linearity: If $f(x) = \underset{\text{dep. on } x_i}{a^{(0)}} u + \sum_{i=1}^{n} a_i^{(1)} u_{x_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(2)} u_{x_i x_j} + \dots$
- Homogeneity: $f(x) = 0$ (so no term not depending on u)
- Quasi-linearity: linear in highest order derivative term
- Semi-linearity: all derivatives are linear (but not u itself)

Superposition principle: if a PDE is linear and homogenious, then any lin. comb. of solns is also a soln.

ODE Solving Methods

linear ODE & coef: $\boxed{a_n y^{(n)}(x) + \dots + a_0 y(x) = f(x)}$, $y(x) = y_h(x) + y_p(x)$

y_h: - solve char (λ) $(y^{(n)} \to \lambda^n) \to \lambda, \lambda_1, \lambda_2, \dots$
- $y_H = Ae^{\lambda_1 x} + (B + Cx)e^{\lambda_2 x} + \dots$

y_p: - $y_p \sim$ like $f(x)$ with a & in front
→ $y = y_H + y_p$

separable ODE: $\boxed{\frac{d}{dx} y(x) = f(x) \cdot g(y)} \to \frac{1}{g(y)}\, dy = f(x)dx$

$G(y) = \int \frac{1}{g(y)} dy = \int f(x) dx = F(x)$, solve for y

Variation of constant: $\boxed{\frac{d}{dx} y(x) = a(x) \cdot y + b(x)}$

$A(x) = \int a(x) dx$

$y(x) = F \cdot e^{A(x)} + P(x) e^{A(x)}$ $\quad P(x) = \int b(x) e^{-A(x)} dx$

2 Method of Characteristics (1st order quasilinear PDEs)

$F(x,y,u,u_x,u_y) = 0$ → establish a relation between the soln u and the tangent plane to the graph u

General form: $\begin{cases} a(x,y) u_x(x,y) + b(x,y) u_y(x,y) = c(x,y,u) \\ u(x_0(s), y_0(s)) = u_0(s) \end{cases}$

1. Find the initial curve from the initial condition
$\Gamma(s) = (x_0(s), y_0(s), \tilde{u}_0(s)) \equiv (x(s,0), y(s,0), \tilde{u}(s,0))$
$x(s,t), y(s,t)$ are the characteristics
2. Solve the following system of ODEs:
$\frac{\partial}{\partial t} x(t,s) = a(x(t,s), y(t,s))$ & $x(0) = x_0(s)$
$\frac{\partial}{\partial t} y(t,s) = b(x(t,s), y(t,s))$ & $y(0) = y_0(s)$
$\frac{\partial}{\partial t} u(t,s) = c(x(t,s), y(t,s), u(t,s))$ & $u(0) = u_0(s)$
3. Express s and t depending on x and y
4. Insert $s(x,y)$ & $t(x,y)$ in $u(t,s)$ and find $u(x,y)$
5. Verify the solution

For the problem to have a soln, it must fulfill the

Transversality conditions: $J = \begin{vmatrix} x_t(0,s) & y_t(0,s) \\ x_s(0,s) & y_s(0,s) \end{vmatrix} = \begin{vmatrix} a & b \\ \partial_s x_0 & \partial_s y_0 \end{vmatrix} \neq 0$

where this is true, there exists a local solution, because the mapping $x = x(t,s)$, $y = y(t,s)$ is invertible

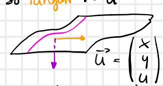
if $= 0$ then there is 0 or ∞-many soln.

Obstacles to global existence:

i) Solutions can blow up in finite time
ii) If the characteristics cross the initial curve $\Gamma(s)$ more than once
iii) If characteristics intersect with each other

Graphical interpretation, we can rewrite the problem as:

$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} u_x \\ u_y \\ -1 \end{pmatrix} = 0$ normal to solution plane $\vec{u}(x,y)$ so $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is ⊥ to the normal so tangent to $\vec{u}$



By integrating the tangent over t we get the solution (since tangent is derivative to $\vec{u}$). So we get the following diff. eqn:

$\frac{\partial}{\partial t}\begin{pmatrix} x \\ y \\ u \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ → We get the charact. which are tangent to $\vec{u}$. We can then knit the surface of $\vec{u}$, with the help of I.C.

Conservation law

PDEs describing evolution of conserved quantities (where x spatial, y temporal, $y > 0$)

General Formulation: $u(x,y)$ $\quad u: \mathbb{R} \times [0,\infty) \to \mathbb{R}$

$\boxed{\begin{aligned} (1)\ & u_y + F(u)_x = 0 \\ (2)\ & u_y + c(u) u_x = 0 \end{aligned}}$ equivalent $\quad F: \mathbb{R} \to \mathbb{R}$ flux, $c(u) = F'(u)$

with initial data $u(x,0) = h(x)$

Transport eqn.: $u_y + c\, u_x = 0$ special case with $c \in \mathbb{R}$

We use M.o.C to solve this problem. Characteristics are straight lines. Transversality condition will guarantee local existance and uniqueness up until the critical time:

$$\boxed{y_c = \inf_{s \in \mathbb{R}: c(u_0(s))_s < 0} \left\{ \frac{-1}{c(u_0(s))_s} \right\}}$$

and $y_c = +\infty$ if $c(u_0(s))_s \geq 0$

then $u(x,y) = u_0(x - c(u(x,y))\, y)$ (implicit eqn)

After y_c we must introduce weak solns that satisfies the PDE in each Reagion D_i ($D = \bigcup_{i=1}^{n} D_i$) and the integral form of the PDE (I) in the whole domain D. Boundaries between D_i are shock waves.

(I) $\quad \int_{x_1}^{x_2} \left[u(x,y) \right]_{y_1}^{y_2} dx = -\int_{y_1}^{y_2} \left[f(u(x,y)) \right]_{x=a}^{x=b} dy$

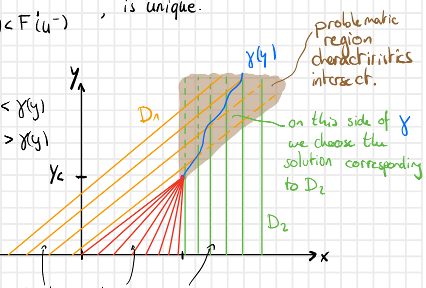
Rankine-Hugoniot: $\boxed{\gamma'(y) = \frac{F(u^+) - F(u^-)}{u^+ - u^-}}$ ≙ speed of shockwave

A weak solution that satisfies the entropy condition:

$\begin{cases} c(u^+) < \gamma'(y) < c(u^-) \\ F'(u^+) < \gamma'(y) < F'(u^-) \end{cases}$, is unique.

$u = \begin{cases} u_{D_1} & x < \gamma(y) \\ u_{D_2} & x > \gamma(y) \end{cases}$

(solves (I))



different characteristics depending on the initial data in each region.

3: 2nd Order linear PDEs

General formulation:

$$\boxed{L\{u\} = a\,u_{xx} + 2b\,u_{xy} + c\,u_{yy} + d\,u_x + e\,u_y + f\,u = g}$$

where a, b, c, d, e, f, g can depend on x and y

- Principal part
- We can describe a 2nd ord. lin. as an Operator $L\{u\}$

Classification: $\delta(L) = b^2 - ac$

- ↳ Hyperbolic $\delta(L) > 0$
- ↳ Parabolic $\delta(L) = 0$
- ↳ Elliptic $\delta(L) < 0$

Homogeneous Wave equation

Homogeneous, one dimensional wave eqn. is a 2nd order diff. eqn.

(1) $\boxed{u_{tt} - c^2 u_{xx} = 0}$ $\quad (x,t) \in \mathbb{R} \times (0, \infty)$

$c \in \mathbb{R}$: wave speed

To solve this eqn we introduce new variables:

$\boxed{\xi(x,t) = x + ct}$ and $\boxed{\eta(x,t) = x - ct}$

$w(\xi, \eta) = u(x(\xi,\eta), t(\xi,\eta))$

Recall: $w(\xi(x,t), \eta(x,t))_t = w_\xi \frac{\partial \xi}{\partial t} + w_\eta \frac{\partial \eta}{\partial t}$

$$\left.\begin{array}{l} u_{tt} = c^2(w_{\xi\xi} - 2w_{\xi\eta} + w_{\eta\eta}) \\ u_{xx} = w_{\xi\xi} + 2w_{\xi\eta} + w_{\eta\eta} \end{array}\right\} \to (1):\ 0 = u_{tt} - c^2 u_{xx} = -4c^2 w_{\xi\eta}$$

$\Rightarrow \boxed{w_{\xi\eta} = 0}$: Canonical form for the wave eqn.

This implies separation of variables: $w(\xi, \eta) = F(\xi) + G(\eta)$

so: $u(x,t) = \underbrace{F(x+ct)}_{\text{Backward wave}} + \underbrace{G(x-ct)}_{\text{Forward wave}}$ (2)

F is ¢ along $x+ct$ ↘ these are characteristics
G is ¢ along $x-ct$ ↗

Cauchy Problem:

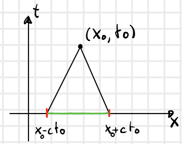
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty) & \text{homo. wave eqn.} \\ u(x,0) = f(x) & x \in \mathbb{R} & \text{initial position} \\ u_t(x,0) = g(x) & x \in \mathbb{R} & \text{initial velocity} \end{cases}$$

Solution is given by: d'Alembert Formula

$$\mapsto \boxed{u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\,dy}$$

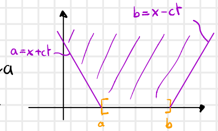
- Singularities propagate along the characteristics.
- If g & f are odd (even) then u is also odd (even).
- So if a problem is well posed only for some x, we can odd- or even-extend it and solve with the standard d'Alembert.

Domain of dependance



The solution in (x_0, y_0) depends on $f(x_0 + ct_0), f(x_0 - ct_0)$ and g in the interval $[x_0 - ct_0, x_0 + ct_0]$

Region of influence



All points satisfying $x - ct \leq b$, $x + ct \geq a$ are dependant on the initial condition on the interval $[a, b]$

Inhomogeneous wave eqn

Cauchy Problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x,t) & (x,t) \in \mathbb{R} \times [0,\infty) \\ u(x,0) = f(x,t) & x \in \mathbb{R} \\ u_t(x,0) = g(x,t) & x \in \mathbb{R} \end{cases}$$

$F(x,t)$ is the external forces acting on the wave

There are two ways to solve this:

1) - Finde/Guess a $v(x,t)$ such that $v_{tt} - c^2 v_{xx} = F(x,t)$

- Use superposition for linear PDE. Since u solves the C.P. and v solves $v_{tt} - c^2 v_{xx} = F(x,t)$. We define $w = u - v$:

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x,0) = f(x,0) - v(x,0) \\ w_t(x,0) = g(x,0) - v_t(x,0) \end{cases} \Rightarrow \text{homog. Cauchy Problem.}$$

- Solve for w
- $u = w + v$

2) Use the modified d'Alembert formula:

$$\boxed{u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\,ds + \underbrace{\frac{1}{2c} \int_0^t \int_{x-c(t-r)}^{x+c(t-r)} F(y,r)\,dy\,dr}_{\text{Integration on the Domain of dependance triangle}}}$$

If we want to solve a wave eqn on some domain, we expend the problem and make initial conditions coincide with the initial problem.

Heat equation

General form: Homogeneous 2nd order linear PDE

$$\boxed{u_t - k\,u_{xx} = 0}$$

Cauchy Problem:

$$\begin{cases} u_t - k\,u_{xx} = 0 & (x,t) \in [0,L] \times [0,\infty) \\ u(x,0) = f(x) & \leftarrow \text{Initial data} \\ \text{one of } \begin{cases} u(0,t) = u(L,t) = 0 & \text{Dirichlet} \\ u_x(0,t) = u_x(L,t) = 0 & \text{von Neumann} \\ \text{or mixed} \end{cases} & \leftarrow \text{Boundary condition} \end{cases}$$

Separation of variables

Used to solve linear PDEs. $\boxed{u(x,t) = X(x)\,T(t)}$

For the Heat eqn we produce the following procedure:

i) $XT' - c^2 X'' T = 0$

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda \quad \begin{array}{l} \nearrow 1)\ X'' = -\lambda X \\ \searrow 2)\ T' = -\lambda k T \end{array}$$

ii) Then with the help of Boundary conditions we get a pair of ODE

$$\begin{cases} X''(x) = -\lambda X \\ X(0) = X(L) = 0 \ \text{or} \ X'(0) = X'(L) = 0 \end{cases}$$

$\{ T'(t) = -\lambda k T(t)$

iii) We make a case distinction $\lambda > 0$: $\cos|\sin(\sqrt{\lambda} x)$, $\lambda = 0$: $Ax + B$, $\lambda < 0$: $\cosh|\sinh(\sqrt{-\lambda} x)$

In each case we first solve X:

$$\left.\begin{array}{ll} \text{D.B.C.}: X_n = \sin\left(\frac{n\pi}{L} x\right) & n = 1,2,3,\ldots \\ \text{v.N.B.C.}: X_n = \cos\left(\frac{n\pi}{L} x\right) & n = 0,1,2,3,\ldots \end{array}\right\} \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

Then using λ_n find the corresponding T: $T_n = e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

iv) For each case: multiply solutions together and get a general $u(x,t)$.

D.B.C: $\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$

v.N.B.C: $\boxed{u(x,t) = \frac{1}{2} B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{L} x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}}$

v) We add the solutions from each case together

vi) Using the initial data we determine the coefficients. If initial data isn't of the same form as the general solution (sin & cos) then we Fourier expand the initial data and compare coefficients.

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L} x\right) \Leftrightarrow a_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx$$

$$f(x) = \frac{b_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi}{L} x\right) \Leftrightarrow b_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx$$

We saw the solutions for the heat eqn. we will now look at the solutions of the **Homogeneous wave eqn**:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ \begin{cases} u(0,t) = u(L,t) = 0 & \text{D.B.C.} \\ u_x(0,t) = u_x(L,t) = 0 & \text{v.N.B.C.} \end{cases} \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

D.B.C.:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[A_n \cos\left(\frac{n\pi}{L}ct\right) + B_n \sin\left(\frac{n\pi}{L}ct\right)\right]}$$

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)dx$$

$$B_n = \frac{2}{cn\pi}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right)dx$$

v.N.B.C.:

$$\boxed{u(x,t) = \frac{A_0 + B_0 t}{2} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left[A_n \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{cn\pi}{L}t\right)\right]}$$

$$A_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right)dx$$

$$B_0 = \frac{2}{L}\int_0^L g(x)dx, \quad B_n = \frac{2}{cn\pi}\int_0^L g(x) \cos\left(\frac{n\pi}{L}x\right)dx$$

For the **Inhomogeneous Wave / Heat eqn** we will use the soln for the homogeneous case without solving for $T(t)$. i.e.:

$$u(x,t) = \begin{cases} \sum_{n=1}^{\infty} T_n(t) \sin\left(\frac{n\pi}{L}x\right) & \text{D.B.C.} \\ \sum_{n=0}^{\infty} T_n(t) \underbrace{\cos\left(\frac{n\pi}{L}x\right)}_{X_n(x)} & \text{v.N.B.C.} \end{cases}$$

Then express the inhomogenity in the corresponding basis. And insert:

$$\sum_n T_n'' X_n(x) - c^2 \sum_n T_n X_n''(x) = h(x,t) = \sum_n c_n(t) X_n(x)$$

$$\sum_n T_n(0) X_n(x) = f(x) = \sum_n c_n X_n(x)$$

$$\sum_n T_n'(0) X_n(x) = g(x) = \sum_n c_n X_n(x)$$

$$\Rightarrow \begin{cases} T_n''(t) - c^2 \left(\frac{n\pi}{L}\right)^2 T_n(t) = c_n(t) \\ T_n(0) = \frac{2}{L}\int_0^L X_n(x) f(x) dx \\ T_n'(0) = \frac{2}{L}\int_0^L X_n(x) g(x) dx \end{cases} \text{ solve ODE for each } n$$

If the boundary conditions are non-homogeneous, find a $v(x,t)$ that satisfies the boundary conditions and substract it from the solution.

4. Elliptic equations

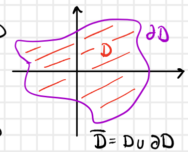
Laplace eqn: $\Delta u = u_{xx} + u_{yy} = 0$

Poisson eqn: $\Delta u = u_{xx} + u_{yy} = f(x,y)$

$u(x,y)$ is a harmonic function if it solves the Laplace eqn

Dirichlet Problem:

$$\begin{cases} \Delta u = f(x,y) & (x,y) \in D \\ u = g(x,y) & (x,y) \in \partial D \end{cases}$$



Neumann Problem:

$$\begin{cases} \Delta u = f(x,y) & (x,y) \in D \\ \partial_\nu u = \vec{\nu} \cdot \vec{\nabla} u = g(x,y) & (x,y) \in \partial D \end{cases}$$

Problem of the third kind:

$$\begin{cases} \Delta u = f(x,y) & (x,y) \in D \\ u + \partial_\nu u = g(x,y) & (x,y) \in \partial D \end{cases}$$

D is a domain in $\mathbb{R}^2$

$$\left.\begin{matrix} \Delta u(x_0,y_0) \leq 0 \\ u_{xx}(x_0,y_0) \leq 0 \\ u_{yy}(x_0,y_0) \leq 0 \end{matrix}\right\} \text{ are conditions for a maximum in } (x_0,y_0)$$

$$\left.\begin{matrix} \Delta u(x_0,y_0) \geq 0 \\ u_{xx}(x_0,y_0) \geq 0 \\ u_{yy}(x_0,y_0) \geq 0 \end{matrix}\right\} \text{ are conditions for a minimum in } (x_0,y_0)$$

$\nabla u(x_0,y_0) = 0$ is a condition for a min/max in (x_0,y_0)

Weak maximum/minimum principle:

Let D be a bounded domain and $u(x,y) \in C^2(D) \cap C(\bar{D})$ a harmonic Function. $u(x,y)$ will take its maximum on ∂D.

$$\boxed{\max_{\bar{D}} u = \max_{\partial D} u} \leftrightarrow \boxed{\min_{\bar{D}} u = \min_{\partial D} u}$$

Proof: let $u_\varepsilon(x,y) = u(x,y) + \varepsilon(x^2+y^2)$ with $\varepsilon > 0$, u harmonic.

if $\max u_\varepsilon(x,y) = u(x_0,y_0) \in D$ then $\Delta u_\varepsilon(x_0,y_0) \overset{!}{\leq} 0$

but on the other hand, $\Delta u_\varepsilon(x_0,y_0) = \Delta u + \Delta \varepsilon(x^2+y^2)\big|_{(x_0,y_0)} = \varepsilon(2+2) = 4\varepsilon > 0$ ↯

so u_ε must obtain its maximum on ∂D.

$$\Rightarrow \max_{\bar{D}} u \leq \max_{\bar{D}} v = \max_{\partial D} v = \max_{\partial D} u + \max_{\partial D} \varepsilon(x^2+y^2)$$

And for $\varepsilon \to 0$ we have $\max_{\bar{D}} u \leq \max_{\partial D} u$ ☒

Mean value theorem

Let $u(x,y)$ be harmonic in D and let $B_R(x_0,y_0) \subseteq D$ be a Ball of radius R centered in (x_0,y_0). Then

$$\boxed{u(x_0,y_0) = \frac{1}{2\pi R}\oint_{\partial B_R(x_0,y_0)} u(x(s),y(s))\,ds = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\,d\theta}$$

Strong Maximum Principle

Let $u(x,y)$ be harmonic in D and u reaches its maximum inside D, then u is constant on all D.

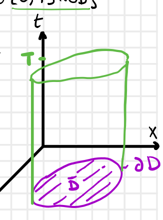
Proof: Use mean value principle and the fact that in a circle around (x_0,y_0), the values must average (x_0,y_0), so they must all be equal to $u(x_0,y_0)$ since $u(x_0,y_0)$ is the maximum. Then the function is ¢ on a circle that we can expand on all D.

Maximum principle for heat equation

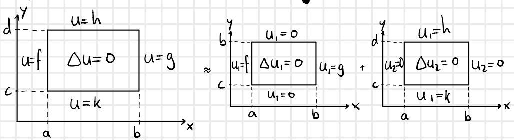
Let $u_t = k\Delta u$ on the domain $Q_T = [0,T] \times D$. The parabolic boundary is defined as: $\partial_p Q_T = \{\{0\} \times D \cup [0,T] \times \partial D\}$

Now, let u be a solution of $u_t = k\Delta u$ on Q_T and let D be a bounded domain. Then u obtains its maximum on $\partial_p Q_T$.

$$\boxed{\max_{Q_T} u = \max_{\partial_p Q_T} u}$$



Separation of variables on rectangular domains

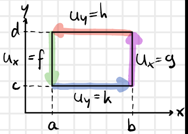


Laplace equation:

General procedure:

- If Neumann B.C., we must verify the following condition:

$$\boxed{\int_{\partial D} \partial_\nu u = 0} = \int_c^d g\,dy - \int_c^d f\,dy + \int_a^b h\,dx - \int_a^b k\,dx$$



- If Dirichlet B.C., the boundary conditions must simply be continuous.
- If needed split the problem in 2 sub-problems, such that boundaries are zero on opposite sides, then verify the conditions from above again. If needed introduce $\tilde{u} = u + \alpha(x^2-y^2)$ and find α such that the condition is verified. Or for D.B.C: we can add a harmonic polynomial $p_H(x,y) = a_0 + a_1 x + a_2 y + a_3 xy$ to u: $\tilde{u} = u + p_H$. Then scale a_i to make the boundaries of $\tilde{u}$ continuous.
- Solve problems for u_1 and u_2 (or $\tilde{u}_1$ and $\tilde{u}_2$), with separation of variables: $u = XY$
  - In the homogeneous direction (x for u_2, y of u_1):

    DBC: $X = A_n \sin(\sqrt{\lambda_n}(x-a))$
    $Y = A_n \sin(\sqrt{\lambda_n}(y-c))$ $\qquad \lambda = \left(\frac{\pi n}{b-a}\right)^2$

    NBC: $X = A_n \cos(\sqrt{\lambda_n}(x-a))$
    $Y = A_n \cos(\sqrt{\lambda_n}(y-c))$
  - In the other direction

    DBC: $Y = C_n \sinh(\sqrt{\lambda_n}(y-c)) + D_n \sinh(\sqrt{\lambda_n}(y-d))$
    $X = C_n \sinh(\sqrt{\lambda_n}(x-a)) + D_n \sinh(\sqrt{\lambda_n}(x-b))$

    NBC: $Y = C_n \cosh(\sqrt{\lambda_n}(y-c)) + D_n \cosh(\sqrt{\lambda_n}(y-d))$
    $X = C_n \cosh(\sqrt{\lambda_n}(x-a)) + D_n \cosh(\sqrt{\lambda_n}(x-b))$

- Use B.C. to find the coefficients
- Add both solutions $u = u_1 + u_2$
- Subtract $\alpha(x^2-y^2)$ if used: $u = \tilde{u} - \alpha(x^2-y^2)$
  p_H ″ : $u = \tilde{u} - p_H$

Laplace equation in Polar coordinates

Laplace operator

$$\boxed{\Delta w = w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta}}$$

Ansatz for circular domains: $w(r,\theta) = R(r)\Theta(\theta)$

$r^2 R''(r) + r R'(r) = \lambda R(r)$

$\Theta''(\theta) = -\lambda \Theta(\theta)$

General solution (for $\Delta w = 0$ and Circle): $\Theta_n(\theta) = A_n \sin(n\theta) + B_n \cos(n\theta)$ $\qquad \sqrt{\lambda_n} = n$

Ansatz for R is r^α, then the ODE becomes:

$r^2 r^{\alpha-2}\alpha(\alpha-1) + r r^{\alpha-1}\alpha = \lambda_n r^\alpha \to \alpha^2 = \lambda_n \to \alpha = \pm\sqrt{\lambda_n}$

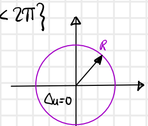
Hence $R_n(r) = \begin{cases} C_0 + D_0 \log(r) & n=0 \\ C_n \cdot r^n + D_n r^{-1} & n \neq 0 \end{cases}$

ignore when origin is contained in D because are singular at the origin

Type 1: Circle $\qquad \bar{D} = \{0 \le r \le R,\ 0 \le \theta < 2\pi\}$

Boundary conditions: $\left.\begin{array}{l}\Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta''(2\pi)\end{array}\right\}$ Periodicity

$w(R,\theta) = f(\theta)$ -> given



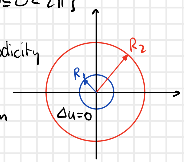
Solution: $\boxed{w(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n (A_n \sin(n\theta) + B_n \cos(n\theta))}$

Insert B.C. and find coefficients.

Type 2: Ring $\qquad \bar{D} = \{R_1 \le r \le R_2,\ 0 \le \theta < 2\pi\}$

Boundary conditions: $\left.\begin{array}{l}\Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta''(2\pi)\end{array}\right\}$ periodicity

$\left.\begin{array}{l} w(R_1,\theta) = f(\theta) \\ w(R_2,\theta) = g(\theta)\end{array}\right\}$ given



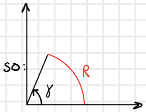
Solution:

$$\boxed{w(r,\theta) = E + F\log(r) + \sum_{n=1}^{\infty} \{ r^n [A_n \sin(n\theta) + B_n \cos(n\theta)] + r^{-n}[C_n \sin(n\theta) + D_n \cos(n\theta)] \}}$$

Type 3: Circle section $\qquad \bar{D} = \{0 \le r \le R,\ 0 \le \theta \le \gamma\}$

Boundary conditions: $w(R,\theta) = h(\theta)$

In this case we only look at N.B.C & D.B.C so:



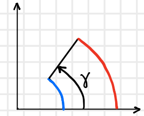
D.B.C. $\Theta(0) = 0$, $\Theta(\gamma) = 0$ $\qquad$ N.B.C. $\Theta'(0) = 0$, $\Theta'(\gamma) = 0$

$w(r,\theta) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}}$ $\qquad$ $w(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}}$

Type 4: Ring section $\qquad \bar{D} = \{R_1 \le r \le R_2,\ 0 \le \theta \le \gamma\}$

Boundary conditions: $w(R_1,\theta) = k(\theta)$
$w(R_2,\theta) = h(\theta)$

In this case we look at N.B.C. and DBC:



D.B.C. $\Theta(0) = 0$, $\Theta(\gamma) = 0$ $\qquad$ N.B.C. $\Theta'(0) = 0$, $\Theta'(\gamma) = 0$

$w(r,\theta) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}}$ $\qquad$ $w(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{\gamma}\theta\right) r^{\frac{n\pi}{\gamma}}$

Then impose the Boundary conditions.

5: General Tipps

- In any exercise with boundary conditions. If we find a function v that solves the problem inside (e.g. $\Delta v = 0$) then we can subtract it from the original problem in u. $w = u - v$. Sometimes, this will simplify the B.C. (e.g. when B.C. are constants, v is also constant)
- To find the whole expression for the shockwave, take the x where the discontinuity is in the initial data.

6: Useful formulas

- $\cos^2(x) + \sin^2(x) = 1$ $\qquad$ · $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- $\frac{\sin k}{\cos k} = \tan k$ $\qquad$ · $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$
- $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$
- $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
- $\int_0^{2\pi} \sin^2(x)\,dx = \int_0^{2\pi} \cos^2(x)\,dx = \pi$

Example of harmonic functions: · 4 $\qquad$ · $1 + \log(x^2 + y^2)$