PVK 2022:

Examples and Exercises

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Example 1.

(Exam Summer 2018)

(a) Solve the characteristic equation in $y \geq 0$

$$\begin{cases} 3uu_x + u_y = u, \\ u(x,0) = x. \end{cases}$$

(i) Check whether the initial condition ~~is non-characteristic.~~ satisfies the transversality condition.

(ii) Solve the characteristic equation.

(i) • We first identify the components of the equation:

$a(t,s) = 3\tilde{u}$, $b(t,s) = 1$, $c(t,s) = \tilde{u}$.

• The next step is to parametrise the initial condition which in this case is relatively simple,

$$\Gamma(s) = (x(0,s), y(0,s), \tilde{u}(0,s))$$
$$= (s, 0, s).$$

So now checking the transversality condition is a simple computation:

$$\begin{vmatrix} a(0,s) & b(0,s) \\ \frac{d}{ds}x(0,s) & \frac{d}{ds}y(0,s) \end{vmatrix}$$

$$= \begin{vmatrix} 3\tilde{u}(0,s) & 1 \\ \frac{d}{ds}(s) & \frac{d}{ds}(0) \end{vmatrix}$$

$$= \begin{vmatrix} 3\tilde{u}(0,s) & 1 \\ 1 & 0 \end{vmatrix}$$

$$= -1.$$

Since this is not zero we conclude that the transversality condition is satisfied.

(ii) Since we have identified

the components of the equation and parametrised the initial curve in part (i), we write down the characteristic equations:

- $x_t = 3\tilde{u}$ $\quad y_t = 1$ $\quad \tilde{u}_t = \tilde{u}$

  $x(0,s) = s$ $\quad y(0,s) = 0$ $\quad u(0,s) = s$.

Since the equation for x depends on $\tilde{u}$ we will first solve for $\tilde{u}$. We easily see that $\tilde{u}(t,s) = se^t$ and so substituting this into the equation for x we find

$$x_t = 3se^t, \quad x(0,s) = s$$

and so $x(t,s) = 3se^t - 2s$. Finally, solving for y we find $y(t,s) = t$. Hence

- $x(t,s) = 3se^t - 2s$, $y(t,s) = t$, $\tilde{u}(t,s) = se^t$.

• We now find the inverse mappings $t = t(x,y)$ and $s = s(x,y)$.
Since $y(t,s) = t$ we can write

$$x = 3se^{y} - 2s \Rightarrow s = \frac{x}{3e^{y} - 2}.$$

So we found that

$$t = y \quad , \quad s = \frac{x}{3e^{y} - 2}$$

and substituting these back into the equation for $\tilde{u}$ we find

$$u(x,y) = s(x,y)\, e^{t(x,y)}$$

$$= \frac{x}{3e^{y} - 2}\, e^{y}.$$

Exercise 1.

(Exam Spring 2018)

(a) Consider the characteristic equation

$$\begin{cases} u_x + xyu_y = xu, \\ u(0,y) = f(y), \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous function.

(i) Check whether the initial condition satisfies the transversality condition.

(ii) Find the explicit formula for u.

Example 2: Consider the transport equation

$$u_y + u^2 u_x = 0$$

with initial data

$$u(x,0) = \begin{cases} 1, & x \leq 0 \\ \sqrt{1-x}, & 0 < x < 1 \\ 0, & x \geq 1 \end{cases}.$$

a) Find the critical time y_c.

b) Find the characteristics and plot them. Show that they intersect at y_c.

c) Find an explicit formula for u for $y \in [0, y_c)$.

d) Extend your solution from part (c) as a weak solution for all $y > 0$.

Solutions:

a) We first identify the flux function as $F(u) = \frac{u^3}{3}$. Hence we can write $c(u) = u^2$ (or just read this off the equation). We also parametrise the initial data as

$$\Gamma(s) = (x_0(s), y_0(s), u_0(s))$$
$$= (s, 0, u_0(s))$$

where

$$u_0(s) = \begin{cases} 1, & s < 0 \\ \sqrt{1-s}, & 0 \leq s \leq 1 \\ 0, & s \geq 1. \end{cases}$$

Now to compute y_c we need to compute the quantity $c'(u_0(s))u_0'(s)$. This comes out to be

$$c'(u_0(s))\,u_0'(s) = \begin{cases} 0, & s<0 \\ -1, & 0\leq s\leq 1 \\ 0, & s>1. \end{cases}$$

From this we easily see that

$$y_c = \inf_{s\in\mathbb{R}:\, c'(u_0(s))u_0'(s)<0} \left\{ -\frac{1}{c'(u_0(s))u_0'(s)} \right\}$$

$$= 1.$$

b) To find the characteristics we begin as in the method of characteristics. Since we already parametrised the initial condition we write down the characteristic equations:

$$x_t = \tilde{u}^2, \qquad y_t = 1, \qquad \tilde{u}_t = 0$$

$$x(0,s) = s, \qquad y(0,s) = 0, \qquad \tilde{u} = u_0(s).$$

Solving these we find that

$y(t,s) = t$ and $\tilde{u}(t,s) = u_0(s)$.

Substituting $\tilde{u}$ in the equation for x we find $x(t,s) = s + u_0(s)^2 y$.

We have three cases for u_0:

$s \leq 0: \quad x = s + y \Rightarrow s = x - y, \; t = y$

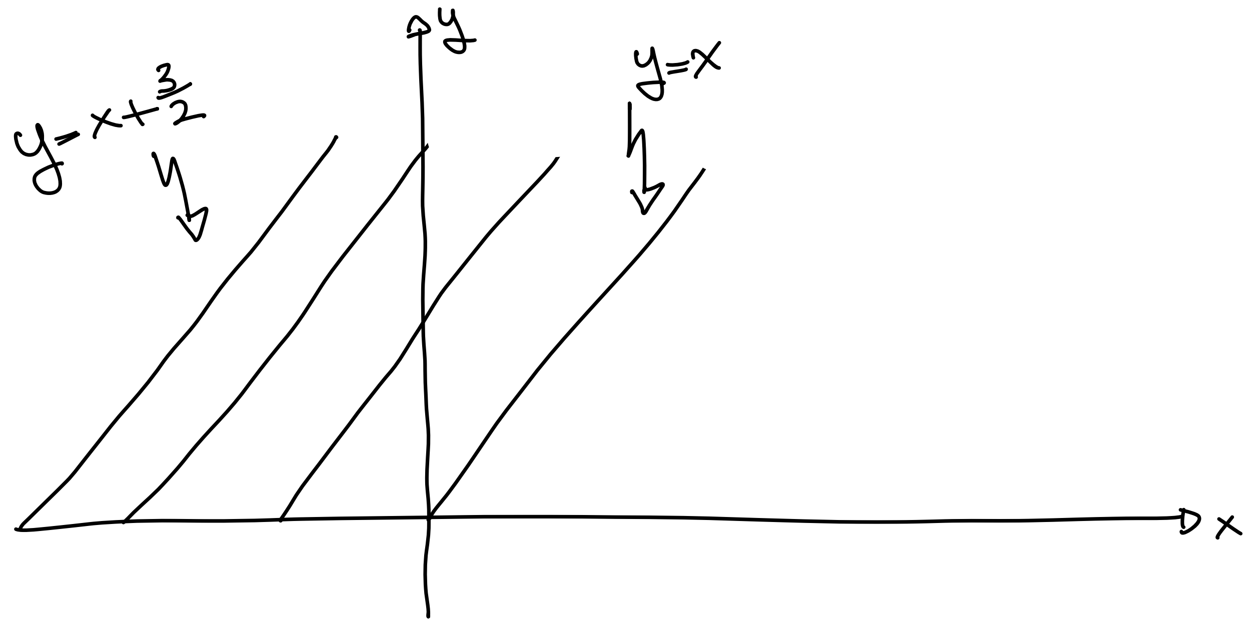
$0 \leq s \leq 1: \quad x = s + (1-s)y \Rightarrow s = \frac{x-y}{1-y}, \; t = y.$

$s \geq 1: \quad x = s + 0 \cdot y \Rightarrow s = x, \; t = y.$

We have now found the characteristics; these are the equations for (s,t). Since $\tilde{u}(t,s) = u_0(s)$, the initial data is <u>transported</u> along the characteristics for s. To see what these look like in the (x,y) variables, we plot the graphs of the characteristics for different values of s.

If $s < 0$, then $s = x - y$

and so we graph for different values of $s<0$ the line $y = x - s$.

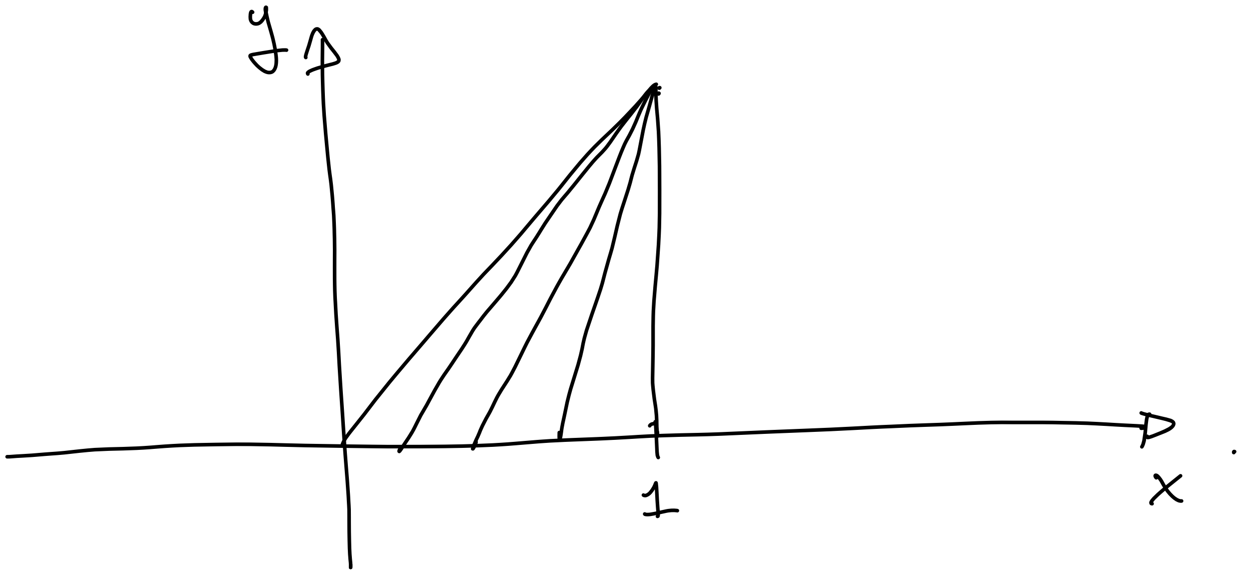


For $0 \leq s \leq 1$ the characteristics are

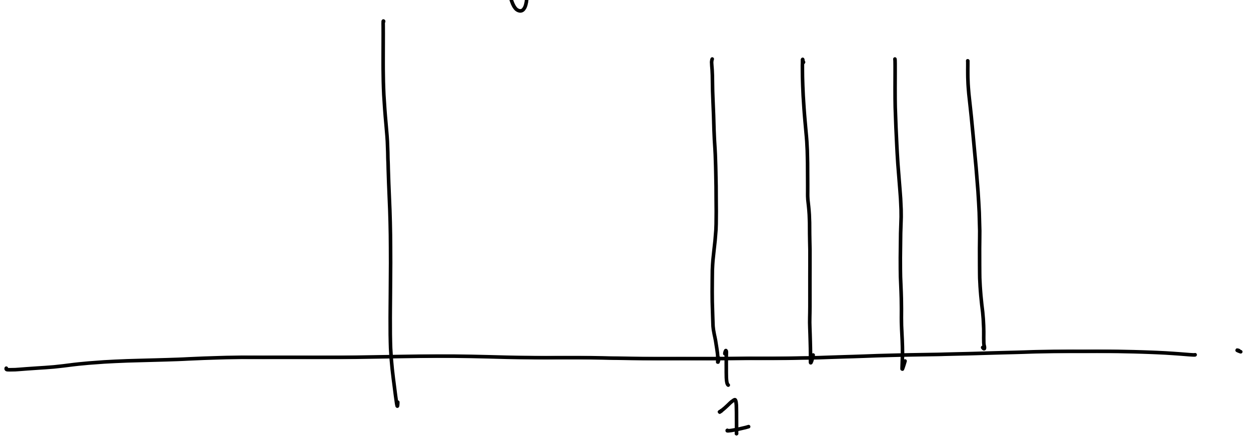
$s = \dfrac{x-y}{1-y}$ and so

$$y = \frac{1}{1-s} x - \frac{s}{1-s} .$$

As $s \to 1$, these lines get steeper and steeper but always pass through the point $(1,1)$ and $(s,0)$. These look like this:



For $s > 1$, the characteristics are simply $x = s$ and plotting these we get



Putting these 3 cases together we obtain a plot of the characteristics up till $y_c = 1$

where it is clear that the characteristics intersect.

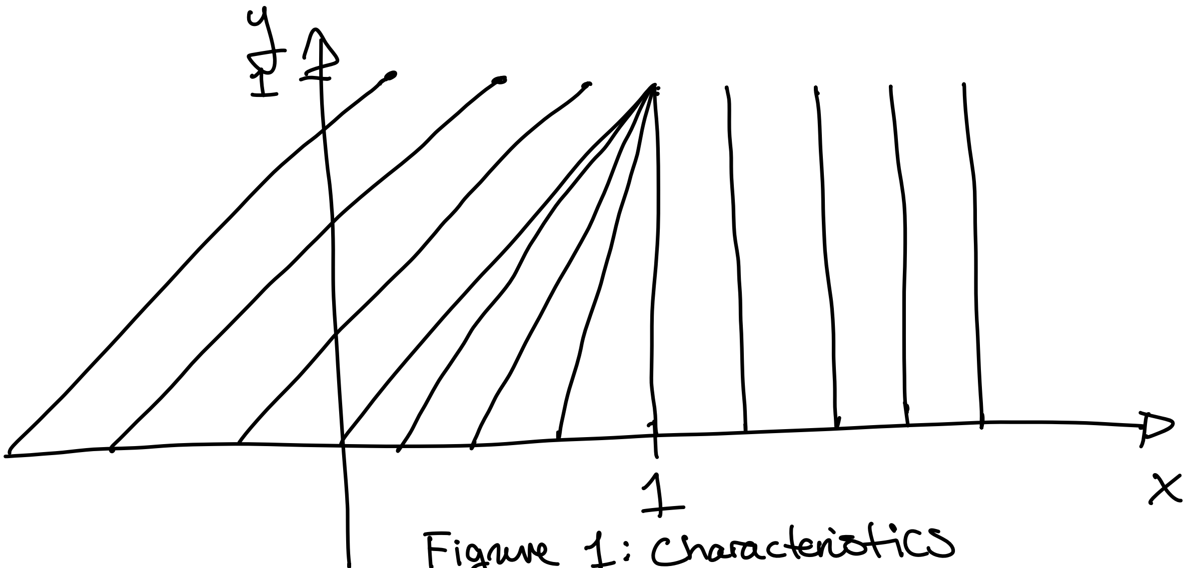


Figure 1: characteristics

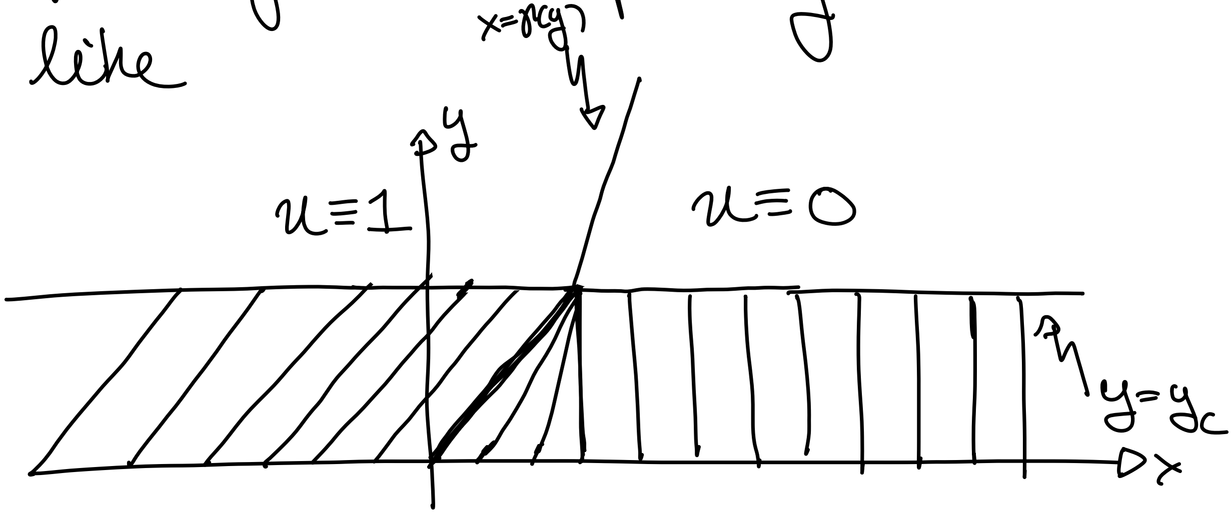
(c) We already solved for the characteristics and that $\hat{u}(t,s) = y_0(s)$, hence we just need to write down the solution:

$$u(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{if } x \geq 1 \\ \sqrt{\frac{1-x}{1-y}} & \text{if } y < x < 1. \end{cases}$$

(d) We look for a weak solution with a shock wave. It is clear from figure 1 that the shock wave will start at (1,1) and so we look for a solution of the form

$$u(x,y) = \begin{cases} 1, & x < \gamma(y) \\ 0, & x > \gamma(y) \end{cases}$$

for $y \geq 1$. Graphically this looks like



To find $\gamma(y)$ we apply the Rankine-Hugoniot condition:

$$\gamma_y(y) = \frac{F(u^+) - F(u^-)}{u^+ - u^-}$$

$$= \frac{1}{3}\,\frac{0 - 1^3}{0 - 1}$$

$$= \frac{1}{3}.$$

Since the shock wave begins at $(1,1)$ we find the equation of the straight line $\gamma(y)$ as

$$\gamma(y) = \frac{1}{3}(y-1) + 1.$$

Hence a weak solution is

$$u(x,y) = \begin{cases} 1 & x \leq y \\ 0 & x \geqslant 1 \\ \sqrt{\frac{1-x}{1-y}} & y \leqslant x \leqslant 1 \end{cases}$$

for $y \in [0, 1)$ and

$$u(x,y) = \begin{cases} 1 & x < \frac{1}{3}(y-1)+1 \\ 0 & x > \frac{1}{3}(y-1)+1 \end{cases}$$

for $y \geq 1$.

Example 3. (Exam Summer 2018)

(b) Consider the scalar conservation law

$$\begin{cases} \frac{\partial}{\partial y}u + \frac{\partial}{\partial x}f(u) = 0, \\ u(x,0) = \begin{cases} 1 & \text{for } x < 0, \\ 3 & \text{for } x > 0. \end{cases} \end{cases}$$

with $f(u) = \frac{1}{3}u^3$.

(i) Determine the characteristics inside the regions

$$\{x < y\} \qquad \text{and} \qquad \{x > 9y\}.$$

(ii) Verify that the following are weak solutions:

$$u_1(x,y) = \begin{cases} 1 & \text{for } x < \frac{7}{3}y, \\ 2 & \text{for } \frac{7}{3}y < x < 4y, \\ \left(\frac{x}{y}\right)^{1/2} & \text{for } 4y < x < 9y, \\ 3 & \text{for } x > 9y, \end{cases} \qquad u_2(x,y) = \begin{cases} 1 & \text{for } x < y, \\ \left(\frac{x}{y}\right)^{1/2} & \text{for } y < x < 9y, \\ 3 & \text{for } x > 9y. \end{cases}$$

(iii) Which of the solutions in part (ii) satisfy the Lax entropy condition? Justify your answer.

(i) We apply the method of characteristics and we obtain that

$$x_t = \tilde{u}^2 \qquad y_t = 1 \qquad \tilde{u}_t = 0$$

$$x(0,s) = s \qquad y(0,s) = 0 \qquad \tilde{u}(0,s) = u_0(s)$$

where $u_0(s) = \begin{cases} 1 & s < 0 \\ 3 & s \geq 0 \end{cases}$.

Hence $y = t$ and $\hat{u}(s,t) = u_0(s)$.

Therefore

$$x_t = u_0(s)^2$$

$$\Rightarrow \quad x = s + u_0(s)^2 t$$

$$= s + u_0(s)^2 y$$

$$= \begin{cases} s + y & s < 0 \\ s + 9y & s > 0 \end{cases}.$$

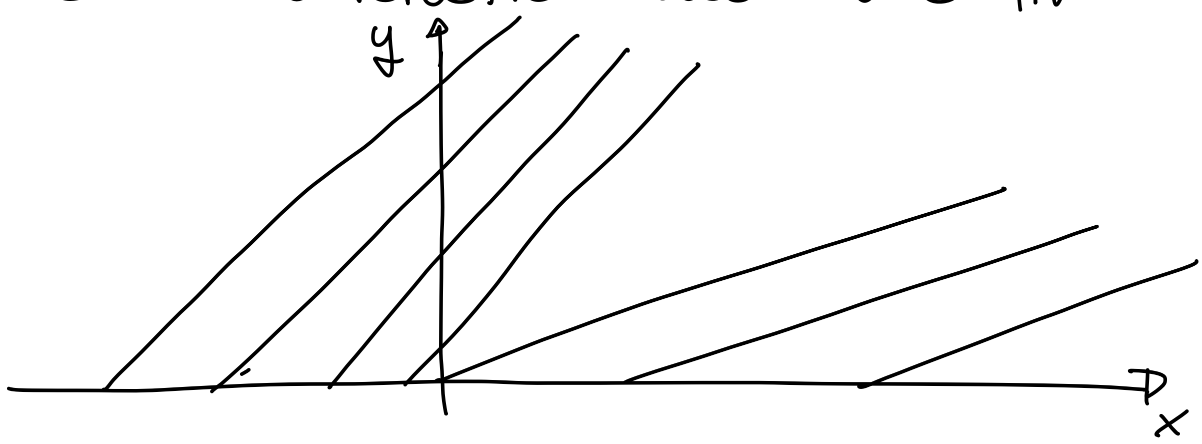
Hence $t = y$, $s = x - y$ for $x < y$,

$t = y$, $s = x - 9y$ for $x > 9y$.

So in $\{x < y\}$, $u(x,y) = u_0(x-y) = 1$

and in $\{x > 9y\}$, $u(x,y) = u_0(x - 9y) = 3$.

The characteristics look like this:



(ii) To show that u_1 is a weak solution we need to first identify where (if any) discontinuities are. By inspection, u_1 is continuous everywhere except across $\{x = \frac{7}{3}y\}$. Now that we have identified the discontinuity, we need to check that

(I) In each region u_1 satisfies the PDE

(II) Across $x = \frac{7}{3}y$ it satisfies the Rankine-Hugoniot condition.

(I) Constants obviously satisfy the PDE and so we just need to check that $\left(\frac{x}{y}\right)^{1/2}$ solves the PDE.

$$\frac{\partial}{\partial y}\left(\left(\frac{x}{y}\right)^{1/2}\right) + \left(\frac{x}{y}\right)\frac{\partial}{\partial x}\left(\frac{x}{y}\right)^{\frac{1}{2}}$$

$$= -\frac{1}{2}x^{\frac{1}{2}}y^{-3/2} + \frac{1}{2}\left(\frac{x}{y}\right)y^{-\frac{1}{2}}x^{-1/2}$$

$$= 0.$$

(II) We just check the Rankine-Hugoniot cond:

$$\frac{F(u^+) - F(u^-)}{u^+ - u^-} = \frac{1}{3}\,\frac{2^3 - 1^3}{2 - 1} = \frac{7}{3}$$

and so it is satisfied and u_1 is a weak solution.

To see that u_2 is a weak solution

we again first check for any discontinuities. However, u_2 is continuous everywhere and so we just need to check that each of the pieces of u_2 solves the PDE, and this is exactly as before.

(iii) First note that since u_2 has no shocks it trivially satisfies the entropy condition.

We check the entropy condition for u_1:

$$c(u_1^+) = 2^2 = 4 > 7/3$$
$$c(u_1^-) = 1^2 = 1 < 7/3 .$$

Hence u_1 does not satisfy the entropy condition.

Exercise 2. (Ex 14.6 2018 Course)

Find a weak solution to the following Cauchy problem with a single discontinuity. Do that by applying the Rankine-Hugoniot condition.

$$\begin{cases} u_y + u^2 u_x = 0 & \text{for } y > 0, x \in \mathbb{R}, \\ u(x,0) = 3 & \text{if } x < 0, \\ u(x,0) = 1 & \text{if } x > 0. \end{cases}$$

Exercise 3. (Exam Spring 2018)

(b)

(i) Let $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be a continuously differentiable function solving the conservation law

$$\frac{\partial}{\partial y} u + \frac{\partial}{\partial x} f(u) = 0, \quad (x,y) \in \mathbb{R} \times [0,\infty),$$

with $f(u) = \frac{1}{3}u^3$. Show that $v = u^2$ solves the conservation law

$$\frac{\partial}{\partial y} v + \frac{\partial}{\partial x} g(v) = 0,$$

with $g(v) = \frac{1}{2}v^2$.

(ii) Let $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be a weak solution to the conservation law

$$\begin{cases} \frac{\partial}{\partial y} u + \frac{\partial}{\partial x} f(u) = 0, & (x,y) \in \mathbb{R} \times [0,\infty), \\ u(x,0) = \begin{cases} 1 & \text{for } x < 0, \\ 0 & \text{for } x > 0, \end{cases} \end{cases}$$

with $f(u) = \frac{1}{3}u^3$.

(1) Find an explicit formula for u.

(2) Is it true that $v = u^2$ is a weak solution of the conservation law

$$\frac{\partial}{\partial y} v + \frac{\partial}{\partial x} g(v) = 0,$$

with $g(v) = \frac{1}{2}v^2$?

(iii) Comparing the results obtained in points (i) and (ii) above, what conclusion can we derive?

Example 4. (2018 Course Ex 7.1)

Consider the initial value problem with zero boundary condition

$$\begin{cases} u_{tt} - u_{xx} &= 0, & (x,t) \in (0,\infty) \times (0,\infty), \\ u(0,t) &= 0, & t \in (0,\infty), \\ u(x,0) &= x^4, & x \in [0,\infty), \\ u_t(x,0) &= \sin(x), & x \in [0,\infty). \end{cases}$$

Evaluate $u(2,1)$ and $u(1,2)$. In which of the two points ($(2,1)$ or $(1,2)$) is the solution unaffected by the boundary condition at $x = 0$?

The first thing to observe here is that we need to solve the wave equation on $x \geq 0$ not $\mathbb{R}$.

Since the D'Alembert formula is derived for $x \in \mathbb{R}$, we need to do some work before we can apply it. Namely, we must define a new problem on $\mathbb{R}$ so that it's solution restricted to $x \geq 0$ yields the desired solution.

Note that if $v(x,t)$ is an odd function then it automatically satisfies $v(0,t) = 0$ for all $t \geq 0$. So in order to satisfy the boundary condition, we extend $u(x,0)$ & $u_t(x,0)$ as odd functions to all of $\mathbb{R}$ and solve

$$\begin{cases} v_{tt} - v_{xx} = 0 & (x,t) \in \mathbb{R} \times (0,\infty) \\ v(x,0) = x^3 |x| & x \in \mathbb{R} \\ v_t(x,0) = \sin(x) & x \in \mathbb{R}. \end{cases}$$

We obtain that $u(x,t) = v(x,t)$ for $x \geq 0$. Now by d'Alembert's formula we get

$$u(x,t) = \frac{1}{2}\left((x+t)^3|x+t| + (x-t)^3|x-t|\right)$$

$$+ \frac{1}{2}\int_{x-t}^{x+t} \sin(s)\,ds$$

$$= \frac{1}{2}\left((x+t)^3|x+t| + (x-t)^3|x-t|\right)$$

$$-\frac{1}{2}\left(\cos(x+t) - \cos(x-t)\right).$$

Now

$$u(1,2) = 40 + \frac{1}{2}\left(\cos(1) - \cos(3)\right)$$

$$u(2,1) = 41 + \frac{1}{2}\left(\cos(1) - \cos(3)\right)$$

We look for the domain of dependence for both points:

$(1,2)$: $(x_0 - Ct_0, x_0 + Ct_0) = (-1, 3)$

$(2,1)$: $(x_0 - Ct_0, x_0 + Ct_0) = (1, 3)$.

Hence, $(2,1)$ is unaffected by the boundary.

Example 5. (Exam Spring 2018)

Let u solve the one dimensional wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = t^2 & \text{for } x \in \mathbb{R},\ t \in \mathbb{R}, \\ u(x,0) = \cos x & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = 0 & \text{for } x \in \mathbb{R}, \end{cases}$$

(i) Find the explicit formula for u.

(ii) Compute $u(0,1)$, $u(0,-1)$, and $u(0,1) - u(0,-1)$.

(iii) Is there a way to compute $u(0,1) - u(0,-1)$ without finding the explicit formula for u?
Hint: look at the equation satisfied by $v(x,t) = u(x,-t)$.

(i) We find first a particular solution $v(x,t) = \frac{t^4}{12}$ and note that $u = v + w \Rightarrow w = u - v$ satisfies

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(x,0) = \cos(x) \\ w_t(x,0) = 0 \end{cases}$$

and so by the D'Alembert formula

$$w(x,t) = \frac{\cos(x+ct) + \cos(x-ct)}{2}.$$

Hence

$$u(x,t) = \frac{\cos(x+ct)+\cos(x-ct)}{2} + \frac{t^4}{12}.$$

(ii) $u(0,1) = \dfrac{\cos(c)+\cos(-c)}{2} + \dfrac{1}{12}$

$u(0,-1) = \dfrac{\cos(-c)+\cos(c)}{2} + \dfrac{1}{12}$

$\Rightarrow \quad u(0,1) - u(0,-1) = 0.$

(iii) We note that $\tilde{u}(x,t) := u(x,-t)$ solves

$$\begin{cases} \tilde{u}_{tt} - c^2\tilde{u}_{xx} = t^2 \\ \tilde{u}(x,0) = \cos(x) \\ \tilde{u}_t(x,0) = 0 \end{cases}$$

and so by uniqueness $\tilde{u} = u$.

Hence

$$0 = u(0,1) - \tilde{u}(0,1) = u(0,1) - u(0,-1).$$

Exercise 4. (2018 Course 7.2)

Solve the following wave equation in the interval $x \in (0,1)$ with zero boundary conditions, for all times $t > 0$. To do so, find a global problem whose solution $\tilde{u}$ coincides with u in the interval $(0,1)$.

$$\begin{cases} u_{tt} - u_{xx} = 0, & (x,t) \in (0,1) \times (0,\infty), \\ u(0,t) = 0, & t \in (0,\infty), \\ u(1,t) = 0, & t \in (0,\infty), \\ u(x,0) = \sin(2\pi x), & x \in (0,1), \\ u_t(x,0) = 0, & x \in (0,1). \end{cases}$$

Exercise 5. Solve for $t>0$:

$$\begin{cases} u_{tt} - u_{xx} = xt, & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = 0, & x \in \mathbb{R}, \\ u_t(x,0) = e^x, & x \in \mathbb{R}. \end{cases}$$

(2018 Course 7.3)

Example 6. Solve the wave equation by separation of variables.

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } [0,\pi]\times(0,\infty) \\ u_x(0,t) = 0 = u_x(\pi,t) & t \geq 0 \\ u(x,0) = 1 + 3\cos(6x) & x\in[0,\pi] \\ u_t(x,0) = \cos(2x) & x\in[0,\pi]. \end{cases}$$

We let $u = X(x)T(t)$ and substituting into the PDE we obtain

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda, \quad \lambda\in\mathbb{R}.$$

Hence for X we obtain

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0, \ X'(\pi) = 0, \end{cases}$$

which has general solution

$$X(x) = \begin{cases} A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x), & \lambda>0 \\ A + Bx, & \lambda = 0 \\ A\cosh(\sqrt{-\lambda}x) + B\sinh(\sqrt{-\lambda}x), & \lambda<0. \end{cases}$$

Now applying the Boundary conditions we see that:

$\lambda < 0: \quad X \equiv 0$

$\lambda = 0: \quad X = A$

$\lambda > 0: \quad B = 0, \quad \lambda_n = n^2, \quad n = 0, 1, \dots$

and so $X_n(x) = A_n \cos(nx)$.

Now we solve for T_n which satisfy

$$T_n'' + n^2 T_n = 0$$

and obtain that

$$T_n = C_n \cos(cnt) + D_n \sin(cnt),$$

for $n \geq 1$ and

$$T_0 = C_0 + D_0 t.$$

We apply the superposition principle and obtain

$$u(x,t) = C_0 + D_0 t + \sum_{n \geq 1} \cos(nx)\left(C_n \cos(cnt) + D_n \sin(cnt)\right).$$

To find C_n, D_n we impose the initial conditions:

$$u(x,0) = \sum_{n \geq 0} C_n \cos(nx) = 1 + 3\cos(6x)$$

$$u_t(x,0) = \sum_{n \geq 0} D_n\, n \cos(nx) = \cos(2x)$$

Hence, we get that

$$C_0 = 1, \quad C_6 = 3, \quad D_2 = 1/2$$

and all the rest are zero. In conclusion we found that

$$u(x,t) = 1 + 3\cos(6x)\cos(6t) + \frac{1}{2}\cos(2x)\sin(2t).$$

Example 7.

$$\begin{cases} u_t - u_{xx} = 1 + x\cos(t), & (x,t) \in (0,1)\times(0,\infty), \\ u_x(0,t) = \sin(t), & t \in (0,\infty), \\ u_x(1,t) = \sin(t), & t \in (0,\infty), \\ u(x,0) = 1+\cos(2\pi x), & x \in [0,1]. \end{cases}$$

We first note that since the boundary conditions are non-homogeneous we must first subtract a function v to "get rid" of them. It is enough to note that

$$v(x,t) = x\sin(t)$$

satisfies the boundary conditions and so $w = u - v$ satisfies

$$\begin{cases} w_t - w_{xx} = 1 \\ w_x(0,t) = 0 \\ w_x(1,t) = 0 \\ w(x,0) = 1+\cos(2\pi x). \end{cases}$$

We solve for w using separation of variables

We could substrate a particular solution and solve the homogeneous problem, however we detail a second method.

Step 1. Express w in the "Fourier basis" associated with the eigenvalue problem with the same boundary conditions. For this we need to solve

$$\begin{cases} X'' + \lambda X = 0 & x \in (0,1) \\ X'(0) = X'(1) = 0 \end{cases}.$$

The solution to this is

$$\lambda_n = \pi^2 n^2, \quad X_n(x) = \cos(n\pi x), \; n = 0, 1, 2, \dots$$

Hence

$$w(x,t) = \sum_{n \geq 0} T_n(t) \cos(n\pi x).$$

Step 2. To find $T_n(t)$ we substitute this expression into the PDE and obtain an ODE for T_n.

Namely, since $w_t - w_{xx} = 1$ we have

$$\sum_{n\geq 0} (T_n' + \pi^2 n^2 T_n(t)) \cos(n\pi x) = 1.$$

Equating co-efficients we obtain

$$\begin{cases} T_0'(t) = 1 \\ T_n'(t) + \pi^2 n^2 T_n(t) = 0, \quad n \geq 1. \end{cases}$$

Step 3. We find the initial values of T_n by using the initial condition.

Since $w(x,0) = 1 + \cos(2\pi x)$ we have

$$\sum_{n\geq 0} T_n(0) \cos(n\pi x) = 1 + \cos(2\pi x)$$

and again by equating co-efficients

we have

$$\begin{cases} T_0(0) = 1 \\ T_2(0) = 1 \\ T_n(0) = 0 \quad n \neq 0, 2. \end{cases}$$

Step 4. Solve the initial value problems for T_n.

In our case we get

$$T_0(t) = 1 + t$$

$$T_2(t) = e^{-4\pi^2 t}$$

$$T_0(t) = 0 \quad n \neq 0, 2.$$

Step 5. Bring it all together.

Hence,

$$w(x,t) = 1 + t + e^{-4\pi^2 t}\cos(2\pi x).$$

Now that we solved for w we have

$$u(x,t) = v(x,t) + w(x,t) = x\sin(t) + 1 + t + e^{-4\pi^2 t}\cos(2\pi x).$$

Exercise 6 (Exam Summer 2018)

Let u solve the wave equation with Neumann boundary conditions

$$\begin{cases} u_{tt} - u_{xx} = \cos x & \text{in } [0,\pi] \times [0,+\infty), \\ u_x(0,t) = 0,\ u_x(\pi,t) = 0 & \text{for } t \geq 0, \\ u(x,0) = 1 & \text{for } 0 \leq x \leq \pi, \\ u_t(x,0) = 1 & \text{for } 0 \leq x \leq \pi. \end{cases}$$

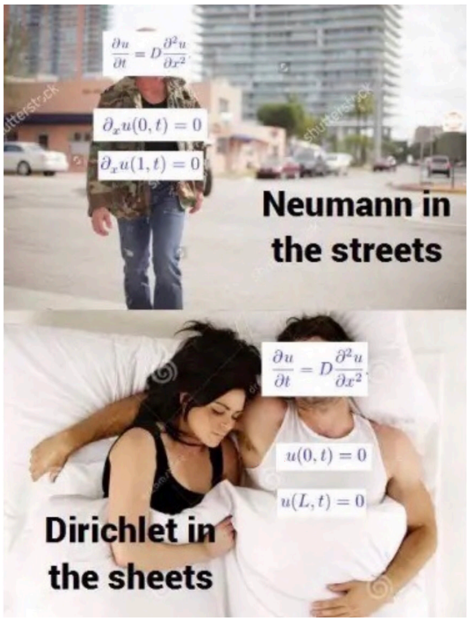
Find the explicit formula for u.

Exercise 7 (Exam Spring 2018)

Let u solve the heat equation with Neumann boundary conditions

$$\begin{cases} u_t - u_{xx} = e^{-2t}\cos x & \text{for } (x,t) \in [0,\pi] \times [0,+\infty), \\ u_x(0,t) = 0,\ u_x(\pi,t) = 0 & \text{for } t \geq 0, \\ u(x,0) = (\cos x)^2 & \text{for } 0 \leq x \leq \pi. \end{cases}$$

Find the explicit formula for u.



Example 8. (Exam Summer 2018)

Consider polar coordinates (r, θ) in $\mathbb{R}^2$, and define the domain $D = \{1 < r < 2, 0 < \theta < \pi\}$. Let $u : D \to \mathbb{R}$ be the solution to the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u(2,\theta) = \sin(\theta) & \text{for } 0 < \theta < \pi, \\ u(1,\theta) = \frac{5}{2}\sin(2\theta) & \text{for } 0 < \theta < \pi, \\ u(r,0) = u(r,\pi) = 0 & \text{for } 1 < r < 2. \end{cases}$$

Find the explicit formula for u.
Hint: use that the functions $\{r^n \sin(n\theta)\}_{n>0}$ and $\{r^{-n}\sin(n\theta)\}_{n>0}$ are harmonic in $\mathbb{R}^2 \setminus \{0\}$, and that they vanish for $\theta = 0$ and $\theta = \pi$.

We apply the method of separation of variables in polar co-ordinates. Writing $u(r,\theta) = R(r)\Theta(\theta)$ we have

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

and we get

$$\begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta(0) = 0 = \Theta(\pi) \end{cases}$$

and hence

$$\lambda_n = n^2, \quad \Theta_n = \sin(n\theta) \quad n = 1, 2, \dots .$$

Then the equation for R is

$$R_n'' + r^{-1} R_n' - n^2 r^{-2} R_n = 0$$

which has solution (not needed to know how to solve this)

$$R_n(r) = C_n r^n + D_n r^{-n}, \quad n \geq 1$$

$$R_0(r) = C_0 + D_0 \log(r).$$

Hence

$$u(r,\theta) = \sum_{n\geq 1} \left(C_n r^n + D_n r^{-n} \right) \sin(n\theta).$$

(In fact we could have started from this point if we used the hint!)

We apply the boundary conditions:

$$u(2,\theta) = \sum_{n\geq 1} \left(C_n 2^n + D_n 2^{-n} \right) \sin(n\theta) = \sin(\theta)$$

$$u(1,\theta) = \sum_{n\geq 1} \left(C_n + D_n \right) \sin(n\theta) = \frac{5}{2}\sin(2\theta)$$

from which we see

$$2C_1 + \frac{1}{2} D_1 = 1, \quad 2^n C_n + 2^{-n} D_n = 0 \quad n \geq 2$$

& $C_2 + D_2 = \frac{5}{2}$, $C_n + D_n = 0 \quad n \neq 2.$

From these we solve for C_1, D_1 as follows:

$2C_1 + \frac{1}{2}D_1 = 1$ & $C_1 + D_1 = 0$

$\Rightarrow D_1 = -2/3$, $C_1 = 2/3$

and simillarly $C_2 = -1/6$, $D_2 = 8/3$.

Putting this all together we obtain

$$u(r,\theta) = \frac{2}{3} r\sin\theta - \frac{2}{3} r^{-1}\sin\theta - 1/6\, r^2 \sin(2\theta) + 8/3\, r^{-2}\sin(2\theta).$$

Exercise 8 (Exam Spring 2018)

Consider polar coordinates (r, θ) in $\mathbb{R}^2$, and define the domain $D = \{1/2 < r < 1, 0 < \theta < \pi\}$. Let $u : D \to \mathbb{R}$ be the solution to the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u(1,\theta) = \sin(\theta) & \text{for } 0 \leq \theta \leq 2\pi, \\ u(1/2,\theta) = 0 & \text{for } 0 \leq \theta \leq 2\pi. \end{cases}$$

Find $u(r,\theta)$.

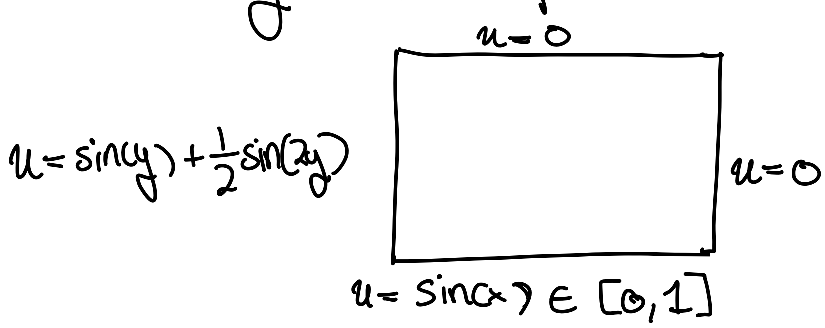
Example 9 Consider

$$\begin{cases} \Delta u = 0 & \text{in } [0,\pi]\times[0,\pi] \\ u(x,0) = \sin(x) \\ u(x,\pi) = 0 \\ u(0,y) = \sin(y) + \frac{1}{2}\sin(2y) \\ u(\pi,y) = 0 \end{cases}$$

a) Is there a point $(x,y) \in (0,\pi)\times(0,\pi)$ so that $u(x,y) = 0$?

b) Solve explicitly for u.

a) We apply the strong maximum principle by first studying what the boundary values for u are.



We need to figure out the range of the function

$$f(y) = \sin(y) + \frac{1}{2}\sin(2y) \quad , \quad y \in [0, \pi]$$

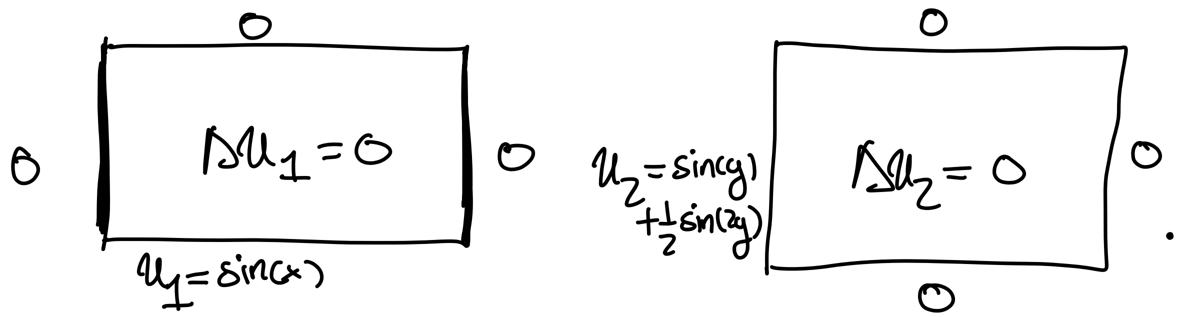
$$= \sin(y) + \sin(y)\cos(y)$$

$$= \sin(y)(1 + \cos(y))$$

and so $f(y) \geq 0$. Hence the minium on the boundary is zero and by the strong maximum principle if there existed an interior point with $u(x,y) = 0$ then u must be constant, $u \equiv 0$. However, clearly the boundary conditions contradict this.

(b) Since we need to have two opposite vanishing boundary conditions to do separation of variables

we split the problem in two.
We set $u = u_1 + u_2$ where



We solve for u_1 and suppose $u_1(x,y) = X(x)\,Y(y)$ and obtain that

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \quad , \lambda \in \mathbb{R}.$$

Which equation to deal with first? The one with the two homogeneous boundary conditions! Note that on the left and right sides we have:

$$u(0,y) = 0 \quad \text{and} \quad u(\pi, y) = 0$$

$$\Rightarrow X(0)\,Y(y) = 0 \quad \text{and} \quad X(\pi)\,Y(y) = 0$$

$$\Rightarrow X(0) = 0 \quad \text{and} \quad X(\pi) = 0.$$

So, we deal with the X equation first and we find:

$$X_n(x) = B_n \sin(nx), \quad \lambda_n = n^2, \quad n = 1, 2, 3 \ldots$$

Then we solve for Y:

$$Y_n'' - n^2 Y_n = 0$$

$$\Rightarrow Y_n(y) = C_n \sinh(ny) + D_n \cosh(ny)$$

$$= C_n \sinh(ny) + \tilde{D}_n \sinh(n(y-\pi))$$

Hence

$$u_1(x,y) = \sum_{n \geq 1} \sin(nx)\left[\alpha_n \sinh(ny) + \beta_n \sinh(n(y-\pi))\right]$$

Now matching BC's we find:

$$u_1(x,0) = \sum_{n \geq 1} \sin(nx)\left[\beta_n \sinh(-n\pi)\right] = \sin(x)$$

$$\Rightarrow \beta_1 = \frac{1}{\sinh(-\pi)}, \quad \beta_n = 0 \quad n \geq 2.$$

$$u_1(x,\pi) = \sum_{n \geq 1} \sin(nx)\left[\alpha_n \sinh(n\pi)\right] = 0 \Rightarrow \alpha_n = 0.$$

Hence,

$$u_1(x,y) = \frac{1}{\sinh(-\pi)} \sin(x)\sinh(y-\pi) .$$

For u_2 we find as above:

$$u_2(x,y) = \sum_{n\geq 1} \sin(ny)\left[\gamma_n \sinh(nx) + \delta_n \sinh(n(x-\pi))\right]$$

and matching boundary conditions:

$$\gamma_n = 0 \quad n \geq 1$$

$$\delta_1 = \frac{1}{\sinh(-\pi)}$$

$$\delta_2 = \frac{1}{2\sin(-2\pi)}$$

$$\delta_n = 0 \quad n \geq 3 .$$

Hence

$$u(x,y) = (\sinh(-\pi))^{-1} \sin(x)\sinh(y-\pi)$$
$$+ (\sinh(-\pi))^{-1} \sin(y)\sinh(x-\pi)$$
$$+ (2\sinh(-2\pi))^{-1} \sin(2y)\sinh(2(x-\pi)) .$$

Example 10. (Exam Summer 2018)

(a) Let $u : D \to \mathbb{R}$ be the solution to the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = 1 - x - y^2 & \text{on } \partial D, \end{cases}$$

where $D = \{x^2 + y^2 < 1\}$.

(i) Compute $u(0,0)$.

(ii) What is the minimum of u?

(b) Let $u : D \to \mathbb{R}$ be the solution to the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = 1 & \text{on } \partial D, \end{cases}$$

where $D = \{x^2 + y^2 > 1\}$ is the complement of the unit disk.

(i) Find two different solutions.

(ii) Does the maximum principle (namely, all solutions attain they maximum on ∂D) hold for this problem?

(c) Let $D \subset \mathbb{R}^2$ be a bounded domain, and let $u(x,y)$ be a solution to the PDE

$$\begin{cases} u_{xx} + y^2 u_{yy} + u_y = 0 & \text{in } D, \\ u(x,y) = f(x,y) & \text{on } \partial D. \end{cases}$$

(i) Prove that u attains its minimum on ∂D.

Hint: consider $v(x,y) = u(x,y) - \epsilon(x^2 + y^2)$ with $\epsilon > 0$, and prove that v can attain a minimum only on ∂D.

(ii) Prove that the solution is unique.

a i) We apply the mean value formula:

$$u(0,0) = \frac{1}{2\pi} \int_{\partial B_1} u(x(s), y(s))\, ds$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(\cos(\theta), \sin(\theta))\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (1 - \cos(\theta) - \sin^2\theta)\, d\theta$$

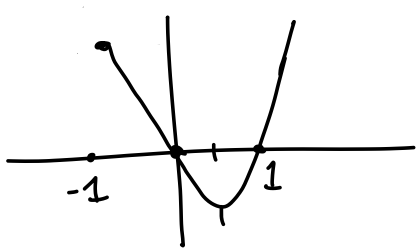
$$= 1 - \frac{1}{2\pi} \underbrace{\int_0^{2\pi} \sin^2\theta \, d\theta}_{\pi}$$

$$= 1/2 .$$

ii) We look for the minimum on the boundary using the parametrisation of the circle:

$$\min_{\partial B_1} 1 - x - y^2 = \min_{\theta \in [0,2\pi)} 1 - \cos(\theta) - \underbrace{\sin^2\theta}_{1-\cos^2\theta}$$

$$= \min_{\theta \in [0,2\pi)} \cos^2(\theta) - \cos(\theta)$$



$$= \min_{x \in [-1,1]} x^2 - x$$

$$= -1/4 .$$

b

(i) $u \equiv 1, \quad v = 1 + \log(x^2 + y^2)$

(ii) Nope! This is the problem with unbounded domains.

C

(i) Consider for any $\varepsilon > 0$

$$v(x,y) = u(x,y) - \varepsilon(x^2+y^2).$$

Suppose there exists $(\bar{x},\bar{y}) \in D$ so that

$$\nabla v(\bar{x},\bar{y}) = 0 \text{ and } D^2 v(\bar{x},\bar{y}) \geq 0.$$

This contradicts the equation since

$$0 \leq v_{xx}(\bar{x},\bar{y}) + \bar{y}^2 v_{yy}(\bar{x},\bar{y}) + v_y(\bar{x},\bar{y})$$

$$= u_{xx} - 2\varepsilon + \bar{y}^2 u_{yy} - 2\bar{y}^2\varepsilon + u_y - 2\varepsilon\bar{y}$$

$$= -2\varepsilon\underbrace{(1+\bar{y}^2-\bar{y})}_{=(\bar{y}-\frac{1}{2})^2+\frac{3}{4}>0.}$$

< 0 ↯.

Hence we obtain

$$\min_{\overline{D}} u \geq \min_{\overline{D}} v = \min_{\partial D} v$$

$$= \min_{\partial D} \left(u - \varepsilon(x^2+y^2)\right)$$

$$\geq \min_{\partial D} u - \varepsilon \max_{\partial D}(x^2+y^2)$$

the letting $\varepsilon \to 0$ concludes the proof.

b(ii) let u_1, u_2 be two solutions.
Observe that $u_1 - u_2$ solves

$$(*) \begin{cases} w_{xx} + y^2 w_{yy} + w_y = 0 & \text{in } D, \\ w = 0 & \text{on } \partial D. \end{cases}$$

Hence by (i) $u_1 - u_2 \geq 0$.
On the other hand, $-(u_1-u_2)$ also solves $(*)$ and so $u_1 - u_2 \leq 0$.
Hence $u_1 = u_2$. ☒

Exercise 9.

(Ex 13.3 from 2018 Course)

(a) Let $u : D \to \mathbb{R}$ be the solution to the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = x^2 - 2 & \text{on } \partial D, \end{cases}$$

where $D = \{x^2 + y^2 < 4\}$.

(i) Compute $u(0,0)$.

(ii) What is the maximum of u?

(b) Let $u : D \to \mathbb{R}$ be the solution to the Laplace equation

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = g & \text{on } \partial D, \end{cases}$$

where $D = \{x^2 + y^2 < 1\}$ and g satisfies $|g| \le 1$.

Assume that $u(0, 1/2) = 1$. What is $u(0,0)$?

(c) Let $D \subset \mathbb{R}^2$ be a bounded domain, $T > 0$, and set $Q_T = D \times [0,T]$. Let $u : Q_T \to \mathbb{R}$ be a classical solution to the PDE

$$\begin{cases} u_t = u_{xx} + x^2 u_{yy} + u_y & \text{for } (x,y,t) \in Q_T, \\ u(x,y,t) = g(x,y,t) & \text{on } \partial D \times [0,T] \\ u(x,y,0) = f(x,y) & \text{on } D, \end{cases}$$

namely, u is twice differentiable with respect to (x,y) in Q_T, once differentiable with respect to t in Q_T, and continuous in $\overline{Q_T}$.

(i) Prove that u attains its minimum on the parabolic boundary

$$\partial_P Q_T = \big(D \times \{0\}\big) \cup \big(\partial D \times [0,T]\big).$$

Hint: consider $v(x,y,t) = u(x,y,t) + \epsilon t$ with $\epsilon > 0$, and prove that v can attain a minimum only on $\partial_P Q_T$.

(ii) Prove that there is at most one classical solution.

Exercise 10. (Exercise 11.2 from 2018 course)

Consider the planar domain D, and let ∂D denote its boundary.

(a) Let v be a smooth function solving $v_{xx} + v_{yy} + xv_x + yv_y > 0$ in D. Show that v has no local maxima in D.

(b) Consider now the Dirichlet problem,

$$\begin{cases} \Delta u + xu_x + yu_y &= 0, \quad \text{in } D, \\ u &= f, \quad \text{on } \partial D, \end{cases}$$

for some given continuous function f. Show that if u is a smooth solution (continuous in $\overline{D}$ and twice differentiable in D), then the maximum of u is attained on ∂D. To do so, take the auxiliary function $v_\varepsilon(x, y) = u(x, y) + \varepsilon x^2$.

(c) Prove that the previous problem has a unique solution.

Solutions to exercises.

1) i) $\Gamma(s) = (0, s, f(s))$

$a = 1$, $b = x(t,s)y(t,s)$, $c = x(t,s)\tilde{u}(t,s)$

$$\begin{vmatrix} a(0,s) & b(0,s) \\ \frac{d}{ds}x(0,s) & \frac{d}{ds}y(0,s) \end{vmatrix} = \begin{vmatrix} 1 & x(0,s)y(0,s) \\ 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 1 \neq 0.$$

ii) $x_t = 1$, $y_t = xy$, $\tilde{u}_t = x\tilde{u}$

$x(0,s) = 0$, $y(0,s) = s$, $\tilde{u}(0,s) = f(s)$.

Solve first for x: $x(t,s) = t$

Substitute this into y equation and solve:

$y_t = ty \implies y(t,s) = se^{t^2/2}$.

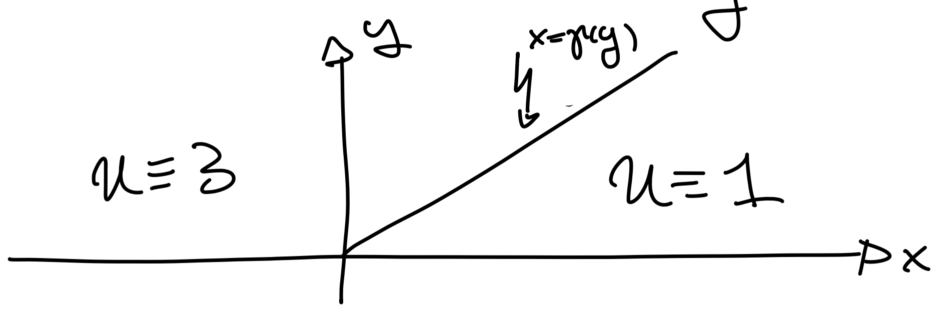
Similarly we solve for $\tilde{u}$:

$$\tilde{u}_t = t\tilde{u} \Rightarrow \tilde{u}(t,s) = f(s)\, e^{t^2/2}.$$

Since $t = x$ we have $s = y e^{-x^2/2}$ and we find

$$u(x,y) = f(y e^{-x^2/2})\, e^{x^2/2}.$$

2) We look for a solution with a shock wave starting at $(0,0)$:



To find $\gamma(y)$ we note:

$$\gamma_y(y) = \frac{F(u^+) - F(u^-)}{u^+ - u^-} = \frac{1}{3}\,\frac{1-3^3}{1-3} = \frac{13}{3}.$$

Hence

$$u(x,y) = \begin{cases} 3 & x < \frac{13}{3} y \\ 1 & x > \frac{13}{3} y. \end{cases}$$

3)

i) We use the chain rule & compute:

$$\frac{\partial}{\partial y} v = \frac{\partial}{\partial y} u^2 = 2u u_y$$

$$\frac{\partial}{\partial x} g(v) = \frac{\partial}{\partial x} \frac{1}{2} v^2 = \frac{\partial}{\partial x} \frac{1}{2} u^4 = 2u^3 u_x$$

and so

$$\frac{\partial}{\partial y} v + \frac{\partial}{\partial x} g(v) = 2u u_y + 2u^3 u_x$$

$$= 2u \underbrace{(u_y + u^2 u_x)}_{=0}$$

$$= 0.$$

ii) 1) The method is same as for exercise (2),

$$u(x,y) = \begin{cases} 1 & x < y/3 \\ 0 & x > y/3 \end{cases}.$$

2) No since it does not satisfy Rankine-Hugoniot.

Indeed

$$v(x,y) = u^2 = \begin{cases} 1 & x < y/3 \\ 0 & x > y/3 \end{cases}$$

and $\dfrac{g(v^+) - g(v^-)}{v^+ - v^-} = \dfrac{g(0) - g(1)}{0 - 1} = \dfrac{1}{2} \neq \dfrac{1}{3}$.

iii) Chain rule is not justified on functions with jumps.

4) We first need to find a problem on $\mathbb{R}$ so that it's restriction to $(0,1)$ is u. We need to extend the initial data oddly around 0 and 1. Since $\sin(2\pi x)$ is odd around 0 and 1 it is enough to solve

$$\begin{cases} \tilde{u}_{tt} - \tilde{u}_{xx} = 0 & x \in \mathbb{R},\ t \geq 0 \\ \tilde{u}(x,0) = \sin(2\pi x) & x \in \mathbb{R} \\ \tilde{u}_t(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

and we apply the d'Alembert formula.

Remark: To see that $\sin(2\pi x)$ is odd around 1 we need to show that

$$\sin(2\pi(1-x)) = -\sin(2\pi(1+x))$$

(i.e. a reflection in the line $x=1$ yields the negative sin graph). But this is a simple computation;

$$\begin{aligned}\sin(2\pi(1-x)) &= \sin(-2\pi x+2\pi)\\ &= \sin(-2\pi x)\\ &= -\sin(2\pi x)\\ &= -\sin(2\pi x+2\pi)\\ &= -\sin(2\pi(x+1)).\end{aligned}$$

▨

5) We subtract the particular solution $v = \frac{xt^3}{6}$ and solve the homogeneous problem

$$\begin{cases} w_{tt} - w_{xx} = 0 \\ w(x,0) = 0 \\ w_t(x,0) = e^x \end{cases}.$$

We obtain

$$u(x,t) = v + w = \frac{xt^3}{6} + \frac{e^{x+t} - e^{x-t}}{2}.$$

6) The solution is

$$u(x,t) = 1 + t + (1 - \cos(t))\cos(x)$$

7) The solution is

$$u(x,t) = \frac{1}{2} + (e^{-t} - e^{-2t})\cos(x) + \frac{1}{2}e^{-4t}\cos(2x).$$

8) $u(r,\theta) = \frac{4}{3} r \sin\theta - \frac{1}{3} r^{-1} \sin\theta$

9)

ai) Apply the mean value principle:

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi} 4\cos^2(\theta) - 2 \, d\theta$$

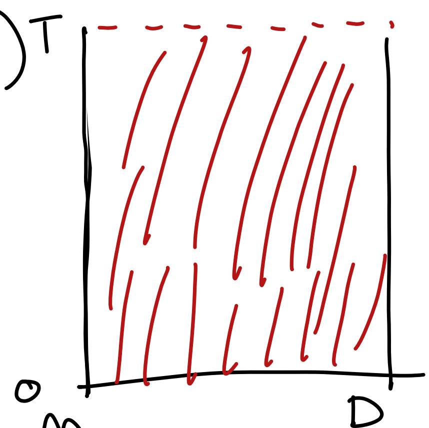
$$= 0.$$

ii) Apply the maximum principle:

$$\max_{\overline{D}} u = \max_{\partial D} u = \max_{\partial B_2} x^2-2 = 2.$$

b) By the strong maximum principle, we have that $u \equiv 1$ since $(0, \frac{1}{2})$ is an interior point and

$$\max_{\partial D} u = \max_{\partial D} g \leq 1.$$

c) 

We must show that u cannot attain it's minimum in the red regions. It will suffice to show that $v = u + \varepsilon t$ cannot attain it's minimum in the region since then we would have

$$\min_{\partial_p Q_T} u \leq \min_{\partial_p Q_T} (u + \varepsilon t) = \min_{Q_T} (u + \varepsilon t) \leq \min_{Q_T} u + \varepsilon T,$$

then letting $\varepsilon \to 0$ we obtain

$$\min_{\partial_p Q_T} u \leq \min_{\overline{Q_T}} u .$$

The inequality $\min_{\partial_p Q_T} u \geq \min_{\overline{Q_T}} u$ is obvious since $\partial_p Q_T \subset \overline{Q_T}$ and hence

$$\min_{\partial_p Q_T} u = \min_{\overline{Q_T}} u .$$

So we show now that

$$\min_{\partial_p Q_T} v = \min_{\overline{Q_T}} v .$$

Suppose that v attained its minimum at $(x_0, y_0, t_0) \in \overline{Q_T} \setminus \partial_p Q_T$, we will have two cases.

If $t_0 = T$ then $\nabla v(x_0, y_0, t_0) = 0$ and $v_t(x_0, y_0, t_0) \leq 0$ while $v_{xx}(x_0, y_0, t_0) \geq 0$ and $v_{yy}(x_0, y_0, t_0) \geq 0$. This contradicts the given PDE at (x_0, y_0, t_0) since

$$v_t = u_t + \varepsilon = u_{xx} + x^2 u_{yy} + u_y + \varepsilon$$
$$= v_{xx} + x^2 v_{yy} + v_y + \varepsilon$$
$$\geq \varepsilon.$$

If $t_0 < T$ then $v_t(x_0, y_0, t_0) = 0$ and the exact same computation yields the contradiction.

ii) To see uniqueness of solutions we let u_1, u_2 be two solutions and note that $w = u_1 - u_2$ satisfies

$$(*) \begin{cases} w_t = w_{xx} + x^2 w_{yy} + w_y & \text{in } Q_T \\ w(x,y,t) = 0 & \text{on } \partial D \times [0,T] \\ w(x,y,0) = 0 & \text{on } D. \end{cases}$$

By part (i) $w = u_1 - u_2 \geq 0$. On the other hand $-w = u_2 - u_1$ satisfies $(*)$ and so $u_2 - u_1 \geq 0$ and so $u_1 = u_2$.

10)

a) Suppose (x_0, y_0) is a local max, then $v_{xx}(x_0,y_0) \leq 0$ & $v_{yy}(x_0,y_0) \leq 0$ and $\nabla v(x_0, y_0) = 0$. Hence

$$v_{xx}(x_0,y_0) + v_{yy}(x_0,y_0) + x_0 v_x(x_0,y_0) + y_0 v_y(x_0,y_0) \leq 0,$$

contradicting the PDE.

b) Note that

$$\Delta v_\varepsilon + x \frac{\partial}{\partial x} v_\varepsilon + y \frac{\partial}{\partial y} v_\varepsilon$$

$$= \Delta u + 2\varepsilon + x(u_x + 2x\varepsilon) + y u_y$$

$$= \Delta u + x u_x + y u_y + 2x^2\varepsilon + 2\varepsilon$$

$$= 2x^2\varepsilon + 2\varepsilon > 0$$

and so by part (a) we have

$$\max_{\bar{D}} v_\varepsilon = \max_{\partial D} v_\varepsilon .$$

Then, as usual, we note that $v_\varepsilon = u + \varepsilon x^2$ and obtain

$$\max_{\overline{D}} u \leq \max_{\overline{D}} v_\varepsilon = \max_{\partial D} v_\varepsilon$$

$$\leq \max_{\partial D} u + \varepsilon \underbrace{\max_{\partial D} x^2}_{\leq M \text{ since } D \text{ is bounded}}$$

and letting $\varepsilon \downarrow 0$ we obtain the result.

C) Identical to (9cii).